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AND

DISCUSSIONS

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AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

TALL BUILDING FRAMES STUDIED BY MEANS OF MECHANICAL MODELS

BY FRANCIS P. WITMER[1], M. AM. SOC. C. E.,

AND HARRY H. BONNER,[2] ESQ.

SYNOPSIS

A study of tall building bents under the action of transverse forces, by the use of a mechanical model, is contained in this paper. Information is lacking as to the actual effects produced upon vertical reactions and direct stresses in columns from such transverse forces when the bent is varied by changing the relative width of bays, or the relative size of girders or columns in different parts of the frame. Such information can be obtained mathematically only by the application of laborious theoretical methods; but by the aid of a mechanical model results may be found quickly and with reliable accuracy so far as they concern the trend of reactions for assumed variations in design. The only theoretical consideration utilized is the well-known Maxwell reciprocal theorem.

The conclusions drawn from the study are believed to be of interest to engineers who are concerned with the designing of tall buildings and may also have practical value in guiding the design of such structures along lines which are more accurately in accord with true behavior than most of the admittedly approximate methods in common use, and without, at the same time, entailing any unnecessary sacrifice in economy of material.

OUTLINE OF THE STUDY

The study herein described was confined to the determination of the vertical reactions from transverse forces acting upon bents of four and of six columns having small connections of a pure bending type, with no diagonal sway-braces or knee-braces.

NOTE.—Discussion on this paper will be closed in April, 1936, *Proceedings*.

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Until quite recently the analysis of such a frame under wind action has generally followed one of two theories: (1) The portal theory; and (2) the cantilever theory. Both are based on fundamental assumptions that are not theoretically correct, but either of them results in satisfactory design for structures of ordinary proportions.

By the portal theory as herein understood the wind shear is assumed to be so distributed over the different columns of the bent that the direct compression in any interior column is equal to the direct tension, so that the resultant direct stresses in interior columns are all equal to zero. The direct wind stress is thus confined to the two exterior columns, being compression in the leeward, and tension in the windward, column.

By the cantilever theory as herein understood the wind shear is assumed to be so distributed that the unit direct stress from wind in any column will be in proportion to the distance of that column from the neutral axis of the bent, columns on the leeward side of this axis being in compression and columns on the windward side in tension. This distribution is analogous to the distribution of fiber stresses in a vertical cantilever beam fixed at the bottom.

Various modifications of these two methods have been devised and used with greater or less success. For analysis on a theoretically correct basis, two principal methods have been available: (1) The slope-deflection method; and (2) the method of moment distribution. In this case, again, modifications have been employed, usually in the nature of approximations for the purpose of abbreviating what otherwise would be a lengthy and laborious operation.

In 1930, Henry V. Spurr, M. Am. Soc. C. E., advanced a method of equal deflections[3], recommended specifically for very tall, narrow towers, in which a more correct theoretical basis is desired than is provided with either the portal or the cantilever method, but where the limitations of the slope deflection and the moment distribution methods, both theoretical and practical, render them undesirable, in the judgment of the designing engineer.

This method is based primarily upon the assumption that vertical reactions and direct column stresses from wind are in accordance with the cantilever theory, but with the further condition that sizes of girders and columns must be adjusted so as to render such an assumption justifiable. It is clear that, in general, this relation of stresses and reactions from wind loads will not be developed automatically and that, therefore, some adjustment of sizes will be required in order to produce the relation assumed.

As the time necessary to perform a complete analysis of a tall tower by any theoretically exact method precludes its application to a considerable number of frames of varying proportions, in order to determine convincingly the actual trend of reactions and stresses, the plan was conceived of determining vertical reactions by means of a mechanical model of simple and economical construction and, at the same time, capable of ready transformation from one set of proportions to another, so that a maximum number of cases might be considered in a minimum time. The scale ratio of model to prototype was 1 to 60 for 12-in. bents and 1 to 120 for 6-in. bents.

[3] "Wind Bracing", by Henry V. Spurr, N. Y., McGraw-Hill Book Co., 1930.

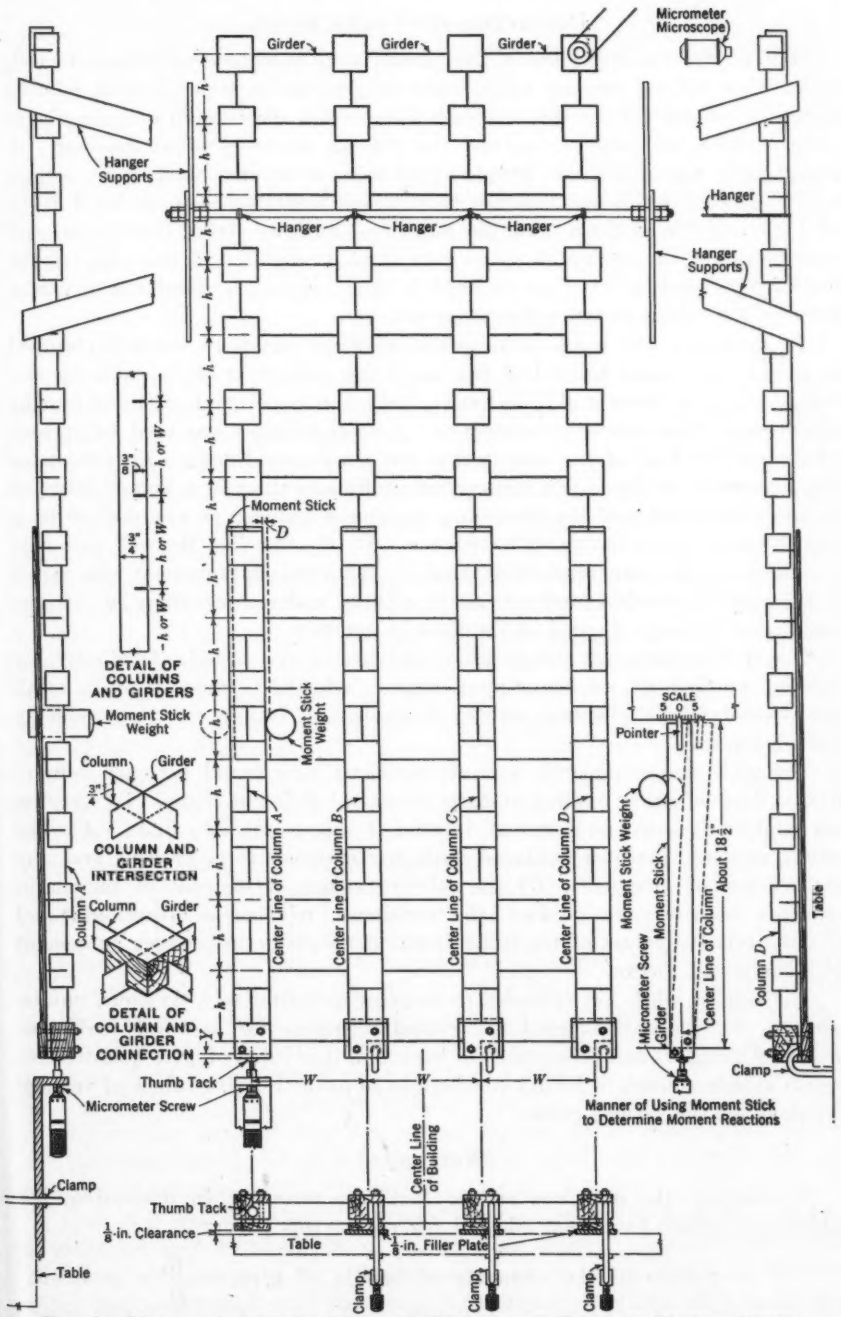

FIG. 1.—MODEL FOR INVESTIGATING THE BEHAVIOR OF BUILDING FRAMES UNDER TRANSVERSE LOADS.

Description and Use of Model

The columns and girders of this model were composed of strips of steel, $\frac{3}{4}$ in. wide and of varying gauge thickness, to produce the desired relative values of moment of inertia. At each intersection the column and the girder were notched half-way across, and the notches mutually fitted together. A gusset-plate connection was simulated by using a wooden block, $\frac{1}{2}$ in. square and 2 in. long, which was notched in two right-angled directions for a depth of $\frac{3}{4}$ in., and forced down upon the joint, one notch receiving the column and and the other the girder so as to hold them firmly out to the edge of the block. The bending was thus confined to the clear length of column or girder between the blocks at two adjacent joints.

In operation, the bases of all columns except one were securely clamped to a table, the model being laid flat and being held clear of the table by two wires suspended from a gallows frame located at a point about three-fourths the distance from the base to the top. A special device was used to prevent rotation at the base of the column that was not clamped down. A micrometer screw was used to move this column longitudinally through a known distance (usually 0.005 in.) and the transverse movement induced at any desired floor was observed by a micrometer microscope. By the familiar Maxwell principle the ratio of the transverse movement to the vertical movement was equal to the vertical reaction produced at the column under observation by a transverse force of unity applied at the floor in question.

Fig. 1 illustrates the construction and use of the model. Obviously, by building up members of one or more properly selected thicknesses of material, any desired relation between values of moment of inertia can be reasonably well obtained.

For each case considered, vertical reactions were found for each column from a load of unity applied at each of several different floors. By varying the relation between moments of inertia, it was possible to find: (*A*) The actual vertical reactions produced with any assumed form of bent and any desired size of members; (*B*) the relative values of moment of inertia in members necessary to produce the cantilever relation of reactions; and (*C*) the relative values of the same function necessary to produce agreement with the portal theory.

The results, while not intended to be exact quantitatively, do show, unmistakably, the qualitative trend of vertical reactions for assumed variations in size of members and proportions of bays and, it is believed, give quantitative results closely enough to justify conclusions as to the relative values of vertical reactions for the various cases.

Notation

To simplify the drawings and to facilitate reference in discussion, the following symbols have been adopted for use in this paper:

i = ratio of the moments of inertia of girders: $\frac{G_i}{G_o}$; as a subscript, i denotes "inner", or "interior bay";

o = a subscript denoting "outer bay";
G = girder; G_i = interior girder; G_o = girder in the outer bays;
I = moment of inertia; I_i = moment of inertia of the girder in the inner bay; and I_o = moment of inertia in the outer bays;
L = center-to-center span length of a girder; L_i = span length of inner girder; L_o = span length of outer girder;
R = column reactions; R_i = average ratio of interior reactions $= \frac{R_B}{R_A} = \frac{R_C}{R_D}$.

Results of Study

The major part of the study dealt with 4-column bents. This type of case was considered in five distinct groups, as follows: Series *A*, *D*, and *E* included bents with three equal bays; Series *B* contained bents with a wide center bay; and Series *C* contained bents with a narrow center bay.

In Series *A*, all bents were 12 in. wide, with three bays of 4 in. each, and with 19 stories of $2\frac{1}{2}$ in. each, making a total height of $47\frac{1}{2}$ in.—a height-to-width ratio of about 4 to 1.

In Case *A*-1 all columns and girders were composed of one strip of steel with a thickness of 0.019 in. The reactions, R_A, R_B, R_C, and R_D, for this case at the four columns, for a horizontal force of unity at the 19th, 15th, 10th and 5th floors, respectively, are shown by curves in Fig. 2(*a*), which indicate such a uniform behavior that interpolated values for other floors may safely be assumed. The ratio of the moment of inertia of the interior girder, G_i, to that of the outer girder, G_o, is denoted by i. By assuming unity at every floor, simultaneously, average reactions were computed from these curves. It was seen that each interior reaction was opposite in character from the exterior reaction on the same side of the bent and was of a relatively small magnitude. Taking the average outer reaction as 100%, the average interior reaction, $R_i \left[= \frac{R_B}{R_A} = \frac{R_C}{R_D} \right]$, was about -15 per cent.

Case *A*-2 was like Case *A*-1, except that all center girders were made of two strips, each 0.019 in. thick, thus making the moment of inertia of this girder twice that of the side girders; that is, $i = 2$. In Fig. 2(*b*) the effect of this change is seen plainly in the reversal in character of the interior reactions, so that each became of the same kind as the outer reaction on the same side of the bent. The average interior reaction, R_i, became about $+33\%$ of the average exterior reaction. This happens to be exactly the relation for the cantilever theory in a bent with three equal bays and with the same size for all columns.

Case *A*-3 was like Case *A*-1, except that all center girders were made of three strips, each 0.019 in. thick, making their moment of inertia three times that of the outer girders. The reactions of inner columns in this case (Fig. 2(*c*)) increased still further to approximate equality with outer reactions, the average value being about $+89\%$ of the latter.

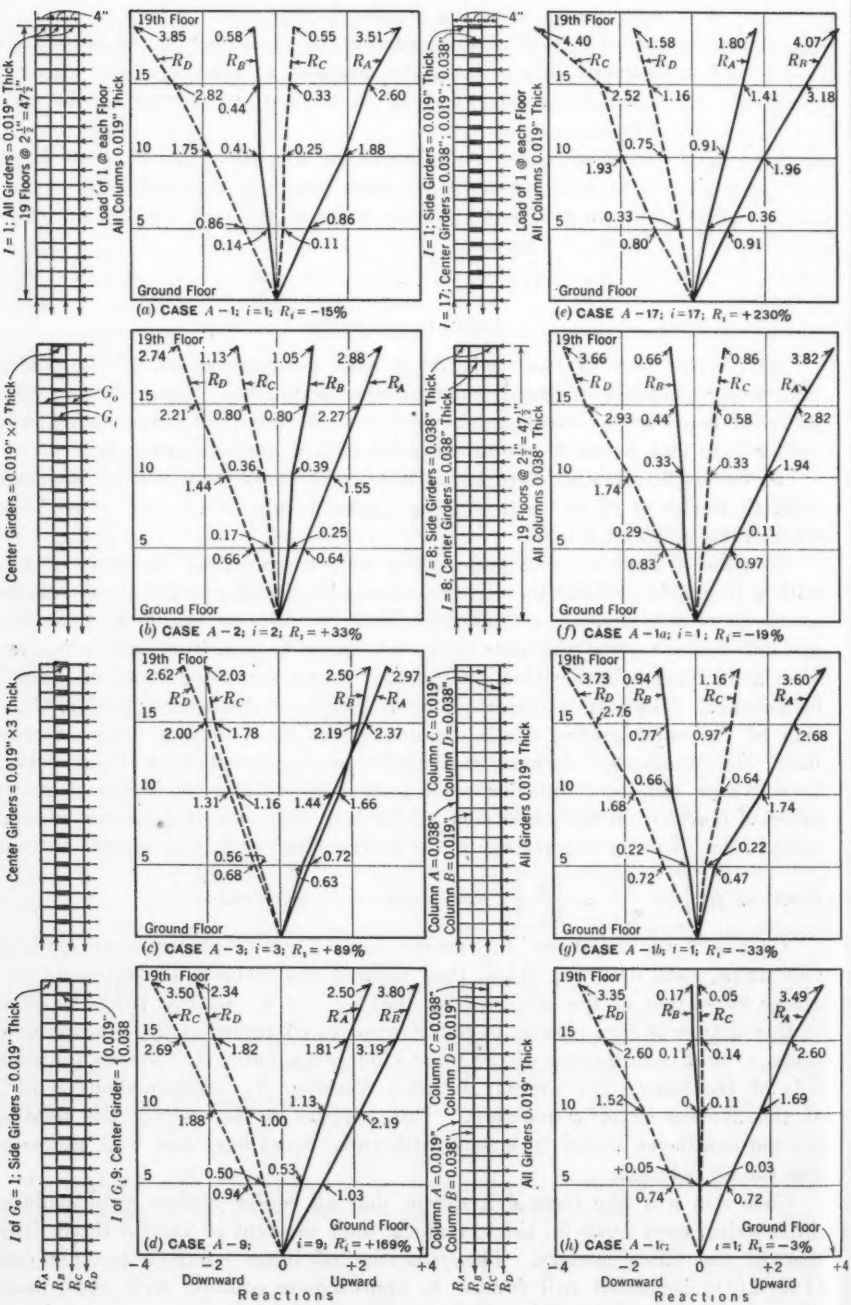
(a) CASE A−1; i=1; R_i=−15%
(e) CASE A−17; i=17; R_i=+230%
(b) CASE A−2; i=2; R_i=+33%
(f) CASE A−1a; i=1; R_i=−19%
(c) CASE A−3; i=3; R_i=+89%
(g) CASE A−1b; i=1; R_i=−33%
(d) CASE A−9; i=9; R_i=+169%
(h) CASE A−1c; i=1; R_i=−3%
19th Floor
Ground Floor
Downward Reactions
Upward Reactions
All Columns 0.019" Thick
All Girders 0.019" Thick

FIG. 2.—REACTION DIAGRAMS, SERIES *A*.

In Case *A*-9 all the center girders were made with one strip 0.019 in. thick, and one, 0.038 in. thick, making their moment of inertia nine times that of the side girders. The interior reactions then were greater than the side reactions, having an average value of about + 169% of the latter (Fig. 2(*d*)).

In Case *A*-17 all center girders were made of two strips 0.038 in. thick, and one, 0.019 in. thick, making their moment of inertia seventeen times that of the side girders (Fig. 2(*e*)). The interior reactions then became more than double the outside reactions, having an average ratio of + 230 per cent.

All the foregoing cases are summarized in Fig. 3, values of i as abscissas being plotted against the ratio, R_i. The points plotted lie on a reasonably regular curve which crosses the abscissa that represents the proper ratio for the cantilever theory relation at a point where $i = 2$. Thus, to obtain the cantilever-theory ratio of reactions, the center girders for a bent of the proportions assumed must be increased until their moment of inertia is about twice that of the side girders. The curve crosses the X-axis at a point corresponding to a moment of inertia of the center girders about one and one-third times that of the side girders, which means that such a relation would produce zero reactions for interior columns and would thus agree with the portal-theory distribution. If all the girders were designed for their vertical loads only, their equality in length of span would lead to a moment of inertia ratio, i, of 1 to 1. This compares quite well with the ratio, $1\frac{1}{3}$ to 1, which was found to produce agreement with the portal-theory reactions.

Three special cases of Series *A* were studied: *A*-1*a*, *A*-1*b*, and *A*-1*c*. In Case *A*-1*a* (see Fig. 2(*f*)), all girders and columns were made 0.038 in. thick instead of 0.019 in. as in Case *A*-1. The results were exactly similar to those of Case *A*-1 with inner column reactions — 19% of the outer reactions, compared with — 15% for Case *A*-1, and again being of opposite character from the outer reactions on the same side of the bent.

In Cases *A*-1*b* and *A*-1*c* (Figs. (2*g*) and 2(*h*)), all the girders were kept 0.019 in. thick as for Case *A*-1, but the columns were varied in thickness. The inner columns remained 0.019 in. thick in Case *A*-1*b*, but the outer columns were made 0.038 in. thick, their area thus being double and their moment of inertia eight times that of the inner columns. The behavior was exactly similar to Cases *A*-1 and *A*-1*a*, except that the value of the inner reaction was increased to — 33% of the outer reaction, remaining opposite in character from the latter on the same side of the bent.

In Case *A*-1*c* the outer columns were made 0.019 in. thick and the inner columns, 0.038 in. thick. The effect was to reduce the inner reactions almost to zero, but they still remained of opposite character from those on the outside of the bent. This relatively great increase in the size of inner columns was thus ineffective in producing the cantilever ratio of reactions, although by the cantilever theory such an increase in the area of inner columns should greatly increase their reactions relative to outer columns and, at the same time, should keep their character the same as that of the latter on the same side of the bent. For the sizes and spacing of columns of Case *A*-1*c* the inner reactions should be + 67% of the outer reactions in order to accord with the cantilever-theory relation, whereas they were — 3% by the model. On the

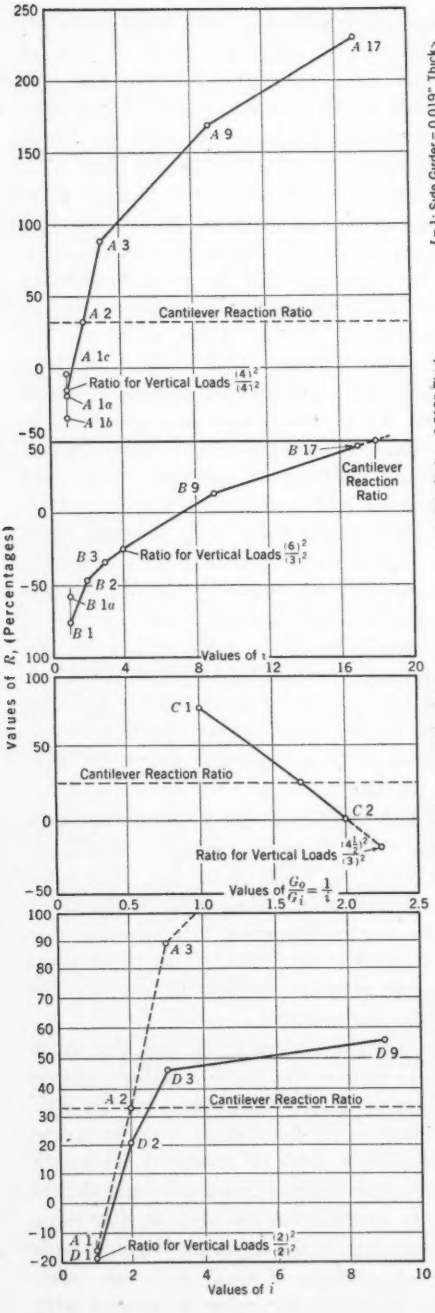

FIG. 3.—SUMMARY, VALUES OF i PLOTTED AGAINST R_i.

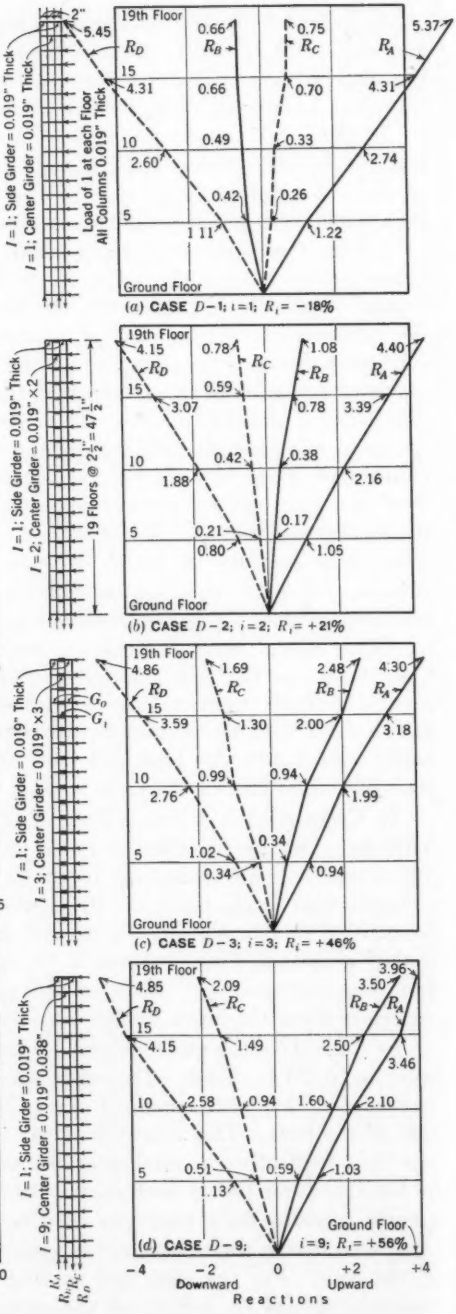

FIG. 4.—REACTION DIAGRAMS, SERIES D.

other hand, the model showed Case *A*-1*c* as agreeing almost exactly with the portal theory, in which inner reactions are assumed to be zero. It was seen by Case *A*-2 that it was only necessary to double the moment of inertia of the center girders to produce the cantilever-theory relation of reactions. It appears, therefore, that, in order to adjust sizes to accord with the cantilever theory, the changes may be made more effectively in girders than in columns.

Series *D* consisted of bents all 6 in. wide, with three equal bays of 2 in. each, and with nineteen stories of $2\frac{1}{2}$ in. each, making the ratio of height to width practically 8 to 1 (see Fig. 4). Case *D*-1, with all girders and columns 0.019 in. thick, agreed very closely with Case *A*-1 for the 12-in. bent having the same sizes. The inner reactions were of opposite character from the outer reactions on the same side of the bent, and their average value was -18% of the outer reactions, as compared with -15% for the 12-in. bent.

Case *D*-2 with center girders of two strips, each 0.019 in. thick, showed behavior similar to Case *A*-2, the inner reaction being changed in character, but its value being only about $+21\%$ of the outer reaction, instead of $+33\%$ as for the 12-in. bent. Cases *D*-3 and *D*-9, with ratios of center girders to side girders, respectively, 3 to 1 and 9 to 1, showed the same trend as Cases *A*-3 and *A*-9, but with relatively smaller average values of the inner reactions.

Fig. 3(*d*) shows that for Series *D* the moment of inertia of the center girders must be about two and one-half times that of the side girders, in order to produce the cantilever relation of reactions. This increase over the required ratio for Series *A* (2 to 1) is doubtless due to the fact that the 6-in. bent, having shorter bending lengths of girders than the 12-in. bent, is relatively more rigid; and, therefore, a greater increase is necessary in the moment of inertia of the center girders, in order to modify the inner reaction sufficiently to produce the cantilever relation.

The *D*-curve crosses the *X*-axis at a point where the ratio, *i*, is $1\frac{1}{2}$ to 1. This compares with $1\frac{1}{3}$ to 1 for the 12-in. bent of Series *A*, and again shows a good agreement with the portal theory, which would require a ratio, *i*, of 1 to 1; but it is a further evidence of the effect of the more rigid 6-in. bent.

In Series *E*, with bents 6 in. wide and with stories the same as for Series *A* and *D*, all the columns were made 0.019 in. thick, but the girders were varied in size, increasing from the top to the bottom of the bent.

In Case *E*-1 (Fig. 5(*a*)) the size of the side girders and the center girders was the same at any floor. Calling the moment of inertia of the top floor unity, the value of *I* for the floors below the top floor increased gradually to a relative value of 8.0 at the bottom. The resulting inner column reactions were practically zero (-2% of the outer reactions), almost an exact agreement with the portal theory.

Case *E*-1.41 (Fig. 5(*b*)) was like Case *E*-1, except that the center girders were increased in such a manner that the total sum of their *I*-values was 1.41 times the sum of the *I*-values for the side girders. The inner column reactions were still very small, but the increase in the center girders was sufficient to reverse their direction and to raise their value to about $+11\%$ of the side column reactions.

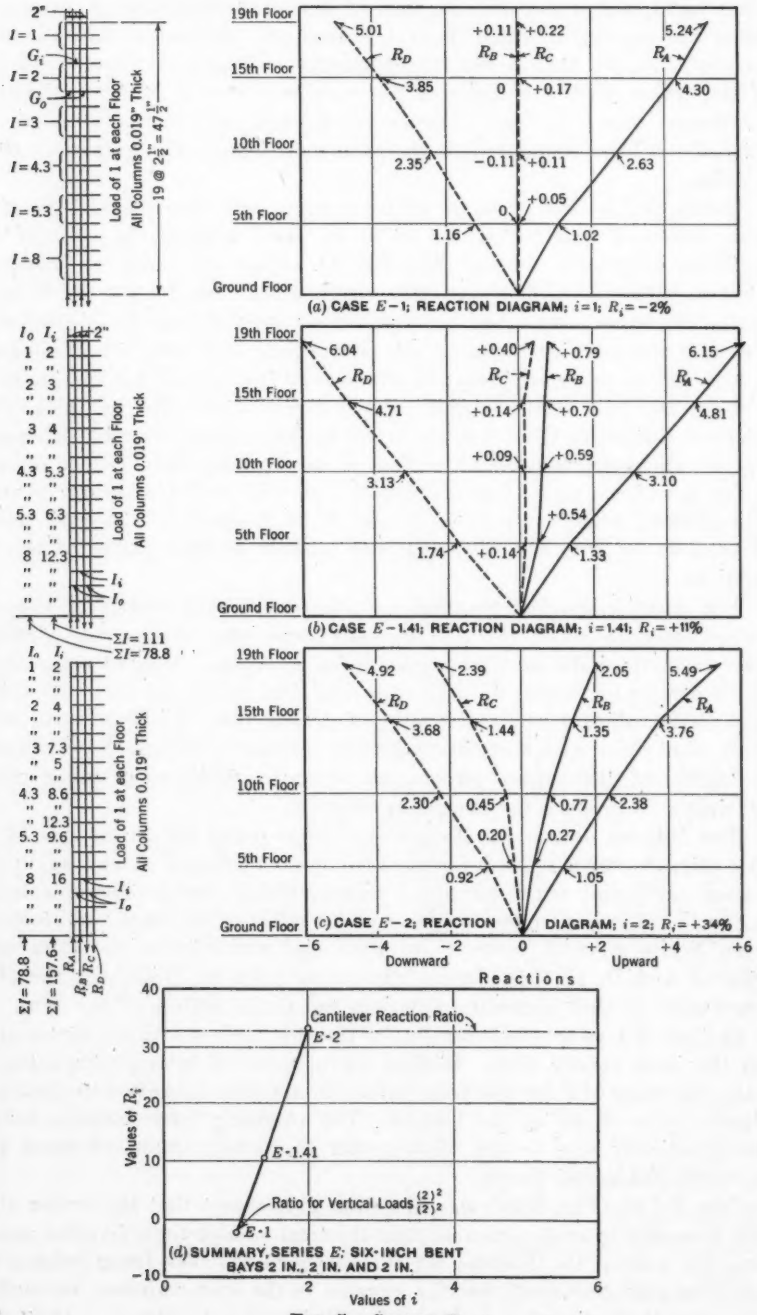

FIG. 5.—SERIES *E*.

Case *E*-2 (Fig. 5(*c*)) was like Cases *E*-1 and *E*-1.41 except that the center girders were increased still further until the sum of all their moments of inertia was double the sum of the *I*-values of the side girders. The value of the inner column reactions was now increased to + 34% of the side column reactions. This agreed with the cantilever theory relation so that, for both Series *A* and Series *E* (see Fig. 5(*d*)), the latter relation required the moment of inertia of the center girders to be about double that of the side girders. For Series *D* (Fig. 3(*d*)) the ratio was about $2\frac{1}{2}$ to 1 instead of 2 to 1—a very fair approximate agreement.

The summaries for Series *A*, *D*, and *E* are strikingly similar in form and reasonably in accord as regards their relation to the cantilever and the portal theory reactions, respectively, being much more nearly in accord with the latter, if the center and side girders are designed for vertical loads alone, and are, therefore, of equal size.

Series *B* (Fig. 6) consisted of bents 12 in. wide, with bays 3 in., 6 in., and 3 in. in width, respectively, and the same story arrangement as for Series *A*, *D*, and *E*.

Case *B*-1 (Fig. 6(*a*)) had columns and girders 0.019 in. thick throughout. The results were similar to those of Case *A*-1, but much more extreme quantitatively. The inner column reactions were again opposite in character from the outer reactions on the same side of the bent, but had a value of − 76% of the latter. Case *B*-2 was like Case *B*-1 except that the center girders had a value of *I* double that of the side girders. Inner reactions had the same character as for Case *B*-1, but were reduced in magnitude to − 46% of the side reactions.

Case *B*-3 was again like Case *B*-1, except that the *I*-value of the center girder was further increased to three times that of the side girder. In this case the inner reactions were further reduced to about − 33% of the outer reactions. Case *B*-9 was the same, but with the ratio, *i*, increased to 9. The inner reactions were changed in character and attained a magnitude of + 13% of the outer reactions. Case *B*-17 was like the preceding cases, but with I_i seventeen times as great as I_o. The inner column reactions became + 46% of the outer reactions. As the cantilever-theory relation for this case was 50%, it would evidently be necessary to increase the ratio, *i*, even more, in order to realize this relation.

Series *B* (Fig. 3(*b*)) shows that the relative *I*-value of the center and the side girders must be about 18 to 1 in order to develop reactions in agreement with the cantilever theory. The curve for this series crosses the *X*-axis at a girder ratio, *i*, equal to about 7 to 1. If designed for vertical loads alone, the relative lengths, 2 to 1, would produce values of *I* in a ratio of about 4 to 1. Although they are not quite in agreement with the portal theory, the reactions for this bent, with girders designed for vertical loads alone, are much more nearly in accord with that theory than with the cantilever theory.

A special case (*B*-1*a*) of this series was considered. In it, columns were kept 0.019 in. thick as for Case *B*-1, but all girders were made 0.038 in. thick. The result was similar to Case *B*-1, but with a reduction in value of the inner column reaction to − 56% of the outer reaction (see Fig. 6(*f*)).

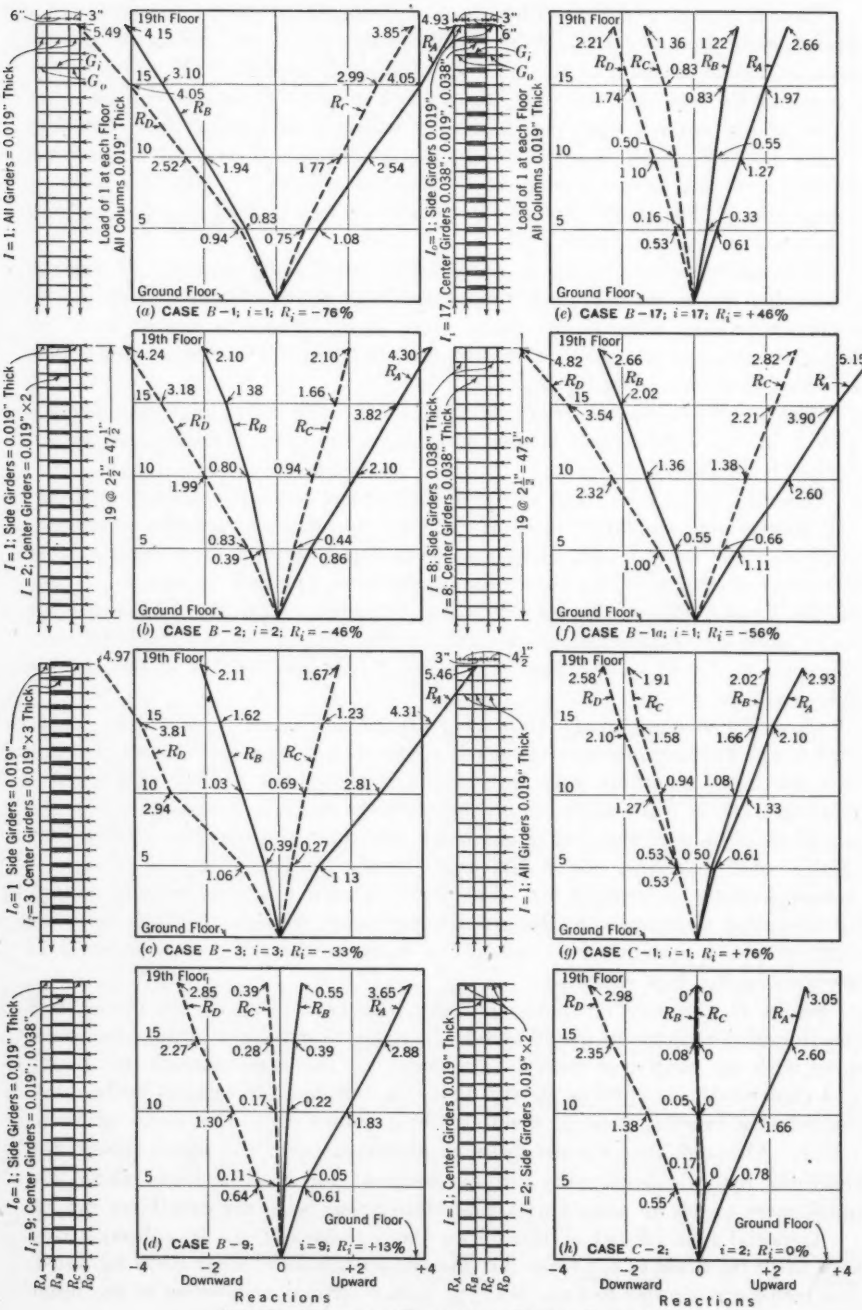

FIG. 6.—REACTION DIAGRAMS; SERIES *B* AND *C*.

Series *C* had 12-in. bents with the same arrangement of floors as for the other four column series (Fig. 6(*g*)) but with bays 4½ in., 3 in., and 4½ in. wide, respectively. Case *C*-1 had all columns and girders 0.019 in. thick. As might be anticipated, the inner column reactions in this case were too large. They had the same character as the outer reactions on the same side, and their value was about + 76% of the latter.

Case *C*-2 was like Case *C*-1, except that the outer girders were increased so that their moment of inertia was twice that of the center girders, their section being two strips, 0.019 in. thick. This reduced the center reactions practically to zero. From Series *C*, Fig. 3(*c*), it appears that the *I*-value of side girders should be about 1.7 times that of center girders to produce the cantilever relation of reactions. If designed for vertical loads alone, the ratio, $\frac{1}{i}$, should be about 2¼ to 1. For agreement with the portal theory the ratio is about 2 to 1 so that, if girders are designed for vertical loads only, the agreement is much more nearly with the portal than with the cantilever-theory relation of reactions.

Summarizing the results of the study of the 4-column bents, the following conclusions seem justified:

(1) With equal bays, all columns being of the same size, the cantilever relation of reactions can only be obtained by making the moment of inertia of the center girders approximately twice that of the side girders.

(2) Increasing the size of either inner or outer columns has relatively less effect upon the reaction ratios than a change in girder ratio.

(3) With a wide center bay, the relative size of center girders, in order to produce the cantilever relation of reactions, is much greater than for equal bays.

(4) With a narrow center bay, I_i must be greater than I_o in order to produce the cantilever-theory reactions.

(5) In any of the foregoing cases if girders are designed for vertical loads only with *I*-values assumed in proportion to the square of the span length, the reactions are more nearly in agreement with the portal theory than with the cantilever theory.

In Fig. 7, Curve *A* shows the reaction ratios $\left(R_i = \frac{R_B}{R_A} = \frac{R_C}{R_D}\right)$ for the cantilever theory and Curve *B* shows the reaction ratios for girders designed for vertical loads only $\left(i = \frac{I_i}{I_o} = \frac{L^2_i}{L^2_o}\right)$. These curves were constructed from Series *A*, *B*, and *C*, as plotted in Fig. 3.

Abscissas are the ratios of inner to outer bay and ordinates are the ratios of inner to outer column reaction, R_i. Apparently, these curves can never intersect. Curve *A* is necessarily always positive and Curve *B* is always negative for the cases considered. Hence, for these cases, if girders are designed for vertical loads only, inner reactions will always be opposite in

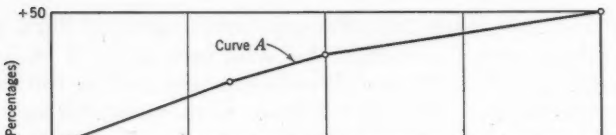

FIG. 7.

character from outer reactions on the same side, and cantilever theory reactions can never be developed except by increasing the moment of inertia, I_i, of the center girders. The increase required for the cases considered, is as follows:

Series	Percentage increase
A	100
B	350
C	33
D	150
E	100

It is seen from this tabulation that Series *C* is naturally most nearly in accord with the cantilever theory.

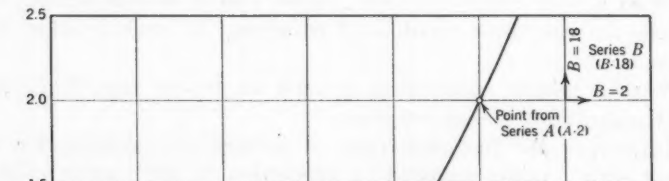

FIG. 8.—CURVE FOR CANTILEVER RATIOS.

Fig. 8 shows the ratio, *i*, necessary to produce cantilever-theory reactions for a varying ratio of bays. It is seen that with a ratio, $\frac{L_0}{L_1}$, of outer bay to

inner bay of about 1.28 to 1, the ratio, i, is 1 to 1. Thus, if the outer bay is about one and one-fourth times the center bay, cantilever reactions will be developed automatically when the girders have equal values of I. To verify the foregoing deduction, a model was made with bays $4\frac{5}{16}$ in., $3\frac{3}{8}$ in., and $4\frac{5}{16}$ in. (ratio, 1.28 to 1), using 0.019-in. thickness for all girders and columns (Case XC-1). The reaction diagram is shown in Fig. 9. The percentage, R_t, of inner reaction to outer reaction was found to be 28.8. The percentage for the cantilever-theory ratio should be 28.0. For vertical loads only the moments of inertia for Case XC-1 should have a ratio, i, of 1.64 to 1. Hence, to produce cantilever-theory reactions the moment of inertia of the center girder would have to be increased 64% above that required for vertical loads only.

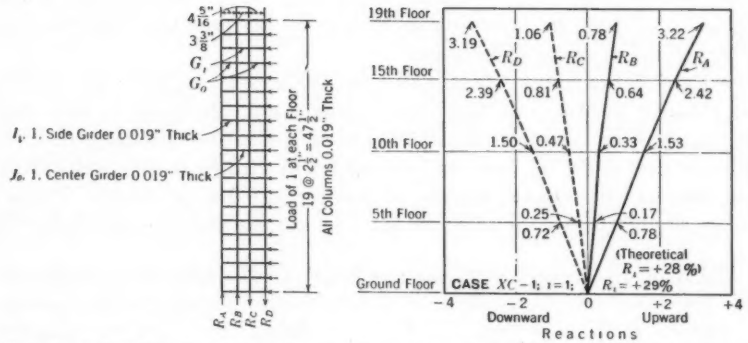

FIG. 9.—REACTION DIAGRAM; CASE $XC-1$; $i=1$; AND $R_t = +29$ PER CENT.

For a model designed in 1934 by the method of equal deflections and tested by Clyde T. Morris, M. Am. Soc. C. E., George E. Large, Assoc. M. Am. Soc. C. E., and Samuel T. Carpenter, Jun. Am. Soc. C. E., the average ratio of experimental moments in the outer and inner girders for the 3d to the 13th floor, inclusive, was found to be about 1.21 to 1, whereas the ratio, i, of actual moments of inertia was 1.88 to 1, the mean of these values being 1.55. The ratio of the areas of the inner and the outer columns in this test model was 1.14 to 1. In Series C, with equal columns, the ratio, i, necessary to produce cantilever-theory reactions is seen (Fig. 3(c)) to be about 1.68 to 1. This is an interesting comparison of the behavior of the two models. The tests were made at the Engineering Experiment Station of the Ohio State University in co-operation with the American Institute of Steel Construction, and the results were published by the Institute in mimeographed form, in November, 1934. There is also a Second Progress Report on this study ("Tests of a Steel Model of a 55-Story Wind Bent"), which was published in mimeographed form, in June, 1935.

STUDY OF SIX-COLUMN BENT

In addition to the foregoing study of 4-column bents, a rather limited study (Series F) was also made of 6-column bents.

In 1933, Professor Morris made a study of a 6-column bent in order to elucidate the method of equal deflections. He used a bent with bays of unequal widths, the center bay being wider than the others and these latter being equal. The columns were of different sizes. A height of about nine times the width of bent was used, 80 stories being assumed, and the girders in the different bays were adjusted in the application of the method of equal deflections in order to accord with the preliminary assumption of the cantilever-theory relation of column reactions.

In the model used for the study of 6-column bents, a width of $8\frac{1}{4}$ in., with bays of $1\frac{9}{16}$ in., $1\frac{9}{16}$ in., 2 in., $1\frac{9}{16}$ in., and $1\frac{9}{16}$ in., was assumed, and a height of $71\frac{1}{16}$ in. in 31 stories of $2\frac{5}{16}$ in. each. These proportions simulated those of the aforementioned 6-column bent, except as to the number of stories. Thicknesses of columns, from outside to center, were, respectively, 0.019 in., 0.026 in., and 0.030 in., agreeing as closely as was practical with the columns in the method of equal deflections.

In Case *F*-1 a single strip, 0.019 in. thick, was used for all girders; in Case *F*-2 the thickness of the center girders was increased to two strips of 0.019 in,; and in Case *F*-3 the thickness of the center girders was increased to three strips, 0.019 in. each, and the adjacent girders to two strips, 0.019 in. thick, leaving the outside girders of one strip, 0.019 in. thick. These sizes were reasonably in accord with the sizes in the method of equal deflections.

TABLE 1.—Reaction Ratios for Six-Column Bent (Percentages)

Series *F*, Case:	Side column	Second column	Inner column
F-1	100	+32	−15
F-2	100	+50	+16
F-3	100	+91	+38
Cantilever-theory relation.....	100	+85	+41

Taking the outside column reactions at 100%, the results of the study for the three cases were as shown in Table 1. It is seen that for Case *F*-1 the inner column reactions are reversed in character from those of the other columns on the same side of the bent. For Case *F*-2, by increasing the girders in the center bay these inner reactions were changed in character and reached a value of + 16% of the outside columns. Case *F*-3, was made to simulate, as closely as practicable, the relation of girders and columns in the method of equal deflections. In this case the percentages for all column reactions agree surprisingly well with those of the cantilever theory, which was a fundamental assumption in the method of equal deflections, and thus verify the correctness of that method of adjustment of girder sizes to accord with the assumption of cantilever-theory reactions.

If girders were designed for vertical loads only, with moments of inertia in proportion to the square of the span length, the proportions of *I*-values for the side girder, second girder, and center girder, respectively, would be as 1:1:1.7. These values lie between the ratios for Cases *F*-1 and *F*-2.

By considering Table 1, the probable relation of column reactions for this relation of moments of inertia in the girders would be about as 100% : 40% : 0%. This seems to indicate that, if girders are designed for vertical loads only, the inner column reactions for this 6-column bent will accord very closely with those of the portal theory and will be quite out of agreement with the cantilever theory.

The following general conclusions appear to be warranted by the foregoing study:

(1) The natural tendency of both 4-column and 6-column bents in which girders are designed for vertical loads only is to develop vertical reactions much more nearly in agreement with the portal theory than with the cantilever theory.

(2) In order to make an assumption of the cantilever-theory distribution of vertical reactions justifiable, it will generally be necessary to increase the moments of inertia of certain girders by an appreciable percentage. For 4-column bents with a wide center bay this increase will be in the center girders and may be prohibitive from a standpoint of economy. For bents with equal bays, the center girder will require a moment of inertia approximately double that which would be required for girders designed for vertical loads only. For bents with a narrow center bay, the side girders must be greater than the center girders, but such bents naturally accord more nearly with the cantilever-theory relation than bents of the foregoing types.

(3) If the assumption of cantilever-theory reactions is desired for any reason, a carefully constructed model should furnish, *a priori*, information as to the probable percentage of increase in the *I*-value of any set of girders in order to develop the assumed relation of reactions.

(4) If a minimum adjustment of girder sizes for wind stress is desired, such a model should furnish a reasonably close relation between the reactions which will actually be developed, and these might be used as the basis of a rational design. This might necessitate assuming that direct stresses in the columns in any story, were in the same proportion as vertical column reactions, but this should not be very far from the truth.

Each column in the foregoing study was always assumed to have the same size from top to bottom. It is not believed that a normal variation in such size, increasing toward the bottom, would greatly vary the general relations discovered and, of course, the sizes of columns in different stories could readily be varied at will in a model of the kind herein described. It is to be remembered that the purpose of the study was to learn trends rather than true quantitative values of reactions for varying proportions of bents. It is thought, however, that, with great care in determining the thicknesses of material to agree with the sizes of an assumed design, and with care in assembling the model, particularly as regards the tightness of fit of joint blocks, reliable quantitative values may be obtained.

Where the model is made with considerable bending length of girders and is, therefore, relatively flexible, the results are most satisfactory. It is found that for such a case the micrometer screw actuates the columns through the

entire distance indicated by the screw, and only the horizontal movement at a floor need be observed with the microscope. If the bent is narrow, with short girders, its greater rigidity may cause a backward movement of the screw where it is clamped so that the actual motion of the column is less than the nominal movement of the screw. In such a case, two microscopes may be used and both the horizontal movement at a floor and the vertical movement of the column observed simultaneously. Their ratio will give the true reaction. It is strongly recommended that models with excessively rigid members be avoided, as liberal a width of bay as practicable being adopted, so that the aforementioned loss will not occur. This will produce results having a maximum degree of reliability.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

MODERN CONCEPTIONS OF THE MECHANICS OF FLUID TURBULENCE

By Hunter Rouse,[1] Assoc. M. Am. Soc. C. E.

Synopsis

Throughout the past century, numerous efforts have been made to understand the problem of resistance in turbulent flow, but only in recent years has a satisfactory analytical basis been laid for such research. The purpose of this paper is to summarize and interpret the modern scientific methods of approach that have made such progress possible, using the subject of flow in circular pipes as the simplest and most pertinent means of illustration.

After a brief discussion of pipe resistance in terms of general dimensionless parameters, the physical nature of turbulence is described qualitatively; the inner mechanism of flow is then followed from the laminar stage through the critical zone and well up into the region of high Reynolds numbers for both smooth and rough pipes. The Prandtl-von Kármán theory of the mixing length, the significance of the laminar boundary layer with relation to surface roughness, the latest experiments on artificial roughness, and von Kármán's universal relationship between the velocity distribution and resistance to flow, are finally treated in terms of the foregoing discussion of the mechanics of fluid motion. The notation used in the paper is summarized in the Appendix.

Dimensional Analysis of Pipe Resistance

For many decades the complexities of turbulent motion in fluids remained so baffling to engineers and scientists alike, despite great progress in other fields of mechanics, that until recent years little had been accomplished toward placing the study of turbulent phenomena upon an analytical basis. As a result, hydraulicians have been forced to rely to a great extent upon empirical relationships for the expression of energy loss due to turbulence; and as empiricism is at best a rather haphazard means to an end, such relationships in themselves are seldom likely to suggest the correct nature of underlying

Note.—Discussion on this paper will be closed in April, 1936, *Proceedings*.

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physical principles. Thus it is that the subject of hydraulics now finds itself somewhat hobbled in its thought by those very methods which contributed greatly to its prestige in past generations.

Hence, it is only logical that the first successful approach to the analysis of turbulence should have come from scientists endeavoring to further the development of a new engineering field. Aeronautics recently found itself confronted with a vast assortment of problems in fluid motion, not the least of which dealt with turbulent formations; and as the new profession was unhampered by tradition, it turned eagerly to the scientific world for much-needed assistance. Astonished as many engineers may have been who had learned to mistrust the ideal notions of classical hydrodynamics, a new and intensely practicable science of fluid mechanics has been developed to meet these needs.

Since turbulence is completely general in its nature, its study through the methods of fluid mechanics is fully as applicable to problems of hydraulics as to problems of airplane design. Those scientists who helped place aeronautics on a sound physical basis have also turned with unabated zeal to the field of pipe flow, and it is only to the advantage of hydraulic engineers that they give serious heed to recent progress in various parts of the world. As the purpose of this paper is to summarize and interpret the most important of these advances in their application to pipe resistance, no attempt has been made at a proper historical treatment; however, abundant references are given in the several texts and papers listed in the footnotes, of which probably the most lucid and complete account is that presented by L. Schiller[2], in 1932.

The first great forward strides were made by Osborne Reynolds in the early Eighties of the Nineteenth Century through his investigations of laminar and turbulent resistance. Although these contributions to the science were forgotten for many years, his discovery of the dimensionless parameter which now bears his name has given the world perhaps its most valuable means of flow analysis. The Reynolds number is one of three dimensionless parameters applicable to the general movement of fluids in which the factor of compressibility—or elasticity—may be ignored[3]. The quality of these numbers is such that any two geometrically similar systems of flow are also dynamically similar if the corresponding dimensionless numbers, named for Froude, Reynolds, and Weber, are numerically equal in each case. Dynamic similarity, in turn, is that condition which prevails if all corresponding forces acting in the two systems are in the same ratio to each other as the inertial qualities of the existing state of motion. That is, according to the Newtonian principle of momentum, the action of any force upon a given mass of matter will produce upon that mass a certain rate of change of momentum:

$$F = \frac{d(m\,v)}{dt} \qquad \text{.....(1)}$$

[2] "Strömung in Rohren", Handbuch der Experimentalphysik, Band IV, 4. Teil, Akademische Verlagsgesellschaft m. b. H., Leipzig, 1932.

[3] "The Analytical Approach to Experimental Hydraulics—On the Use of Dimensional Numbers", by H. Rouse, *Civil Engineering*, November, 1934.

in which F denotes the active force and $\frac{d(m\,v)}{dt}$, the equal and opposite inertial reaction of the mass. In the motion of any fluid, in which the change of density is negligible, only three basic and independent forces—namely, those due to gravity, viscosity, and surface tension—may act to produce such a rate of change of momentum; of these, only the action of viscosity may influence the character of flow entirely enclosed by pipe walls, so that the other two play no further part in this paper. In order for flow through any two geometrically similar pipes of uniform diameter to be dynamically similar, all corresponding viscous forces in the two systems must then bear the same relation to each other as do the inertial characteristics of flow in the two pipes; according to the Newtonian equation, these inertial characteristics must be dimensionally equivalent to the right-hand side of the equation—that is, instead of rate of change of momentum, this inertial term characteristic of uniform flow is the rate of passage of momentum, or the momentum of a unit volume of fluid multiplied by the volume passing a representative cross-section per unit of time. Expressing this inertial quality by the symbol, I, it becomes, in general dimensions of density, velocity, length, and time:

$$I = \rho\, V\, \frac{L^3}{t} = L^2\, \rho\, V^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Similarly, any viscous force, since the intensity of viscous shear equals $\mu\, \frac{dv}{dy}$, must have the following dimensional character:

$$F_v = L^2\, \mu\, \frac{V}{L} = L\, \mu\, V \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The essential equality of force and inertial ratios,

$$\frac{F_{v1}}{F_{v2}} = \frac{I_1}{I_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

then becomes, through substitution of Equations (2) and (3):

$$\frac{L_1\, \mu_1\, V_1}{L_2\, \mu_2\, V_2} = \frac{L^2_1\, \rho_1\, V^2_1}{L^2_2\, \rho_2\, V^2_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and,

$$\frac{L_1\, V_1\, \rho_1}{\mu_1} = \frac{L_2\, V_2\, \rho_2}{\mu_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Thus, in order for dynamic similarity to obtain between the two geometrically similar systems of flow, it is essential that the dimensionless quantity, $\frac{L\, V\, \rho}{\mu}$, be identical numerically in both systems. This quantity, the Reynolds number, is usually given the symbol, R:

$$\mathsf{R} = \frac{L\, V\, \rho}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The length parameter, L, may have any value, characteristic of the flow, which the investigator may care to select, so long as he is consistent in its use throughout the investigation. Obviously, the characteristic length which he selects will determine the final numerical magnitude of the Reynolds number for a given set of conditions; otherwise, the number will be fixed, regardless of whether metric, English, or any other units are used. For purposes of uniformity it is customary, in pipe studies, to use the diameter of the pipe as the length factor.

The Reynolds number is thus seen to represent in simplified form the ratio between the inertial and the viscous properties of flow. If the rate of passage of momentum is increased (by increasing the velocity, density, or pipe diameter), the Reynolds number will grow in magnitude; if the viscosity is increased, the Reynolds number will diminish. As will be shown directly, the larger the value of R, the larger will be the tendency toward instability of flow; conversely, the smaller the value of the Reynolds number, the larger will be the stabilizing effect of viscosity. The magnitude of this number is then a direct indication of the flow characteristics, once a basic value has been established for purposes of comparison. Furthermore, if R has the same numerical value for movement in any two geometrically similar pipes, one may be certain that general conditions of flow in both pipes are identical, regardless of wide variation in diameter, velocity, density, and viscosity between the two systems. It is customary to write the Reynolds number in the form, $\mathsf{R} = \dfrac{V D}{\nu}$; one must remember, however, that the kinematic viscosity, $\nu = \dfrac{\mu}{\rho}$, is the ratio of two independent fluid properties, each of which must be regarded as distinct from the other.

The motion of a fluid passing through a pipe is resisted by a certain tangential stress between the fluid and the pipe walls. Denoting this wall stress per unit area by the customary symbol, τ_0, it is reasonable to presume that the magnitude of the resistance will be a function of pipe diameter, density, viscosity, and average velocity; wall roughness will be omitted for the time being, on the assumption that the pipe is very smooth; that is,

$$\tau_0 = \phi\,(D, \rho, \mu, V) \qquad (8)$$

or, assuming for the present that the function is exponential:

$$\tau_0 = C\ D^x\ \rho^y\ \mu^z\ V^n \qquad (9)$$

Through the principle of dimensional analysis, the form of this function may be found by expressing the terms of Equation (9) in general M-L-T units (M = mass, L = length, and T = time):

$$\frac{M}{L\,T^2} = (L)^x \left(\frac{M}{L^3}\right)^y \left(\frac{M}{L\,T}\right)^z \left(\frac{L}{T}\right)^n$$

The three unknown exponents, x, y, and z, are found in terms of the fourth unknown exponent, n, by the solution of three simultaneous equations, thus:

For M,

$$1 = y + z$$

for L,

$$-1 = x - 3y - z + n$$

and for T,

$$-2 = -z - n$$

Hence: $x = n - 2$; $y = n - 1$; and $z = 2 - n$. By substituting these values for x, y, and z in Equation (9),

$$\tau_0 = C \times D^{n-2} \times \rho^{n-1} \times \mu^{2-n} \times V^n \ldots\ldots\ldots\ldots\ldots (10)$$

which may also be written in the form,

$$\tau_0 = \frac{C'}{\left(\frac{V D \rho}{\mu}\right)^{2-n}} \rho \frac{V^2}{2} \ldots\ldots\ldots\ldots\ldots (11)$$

In Equation (11) the first fraction is seen to be a numerical constant, C', divided by the Reynolds number to the power, $(2 - n)$. This fraction is equal to the customary resistance coefficient, f, divided by the factor, 4; that is,

$$\frac{f}{4} = \frac{C'}{R^{2-n}} \ldots\ldots\ldots\ldots\ldots (12)$$

and, consequently, Equation (11) becomes:

$$\tau_0 = \frac{f}{4} \rho \frac{V^2}{2} \ldots\ldots\ldots\ldots\ldots (13)$$

Inasmuch as the resistance exerted upon any portion of the moving fluid must be counterbalanced by a drop in pressure intensity from one section to the next, the pressure gradient for steady flow in a horizontal pipe of uniform diameter may be written in terms of the total shearing force along the surface of the pipe between the two normal sections:

$$(p_1 - p_2) \frac{\pi D^2}{4} = \tau_0 \pi D L \ldots\ldots\ldots\ldots\ldots (14)$$

and,

$$-\frac{d p}{d L} = \frac{4 \tau_0}{D} \ldots\ldots\ldots\ldots\ldots (15)$$

from which the familiar equation for lost head in pipes may be derived, through substitution of Equation (13):

$$-\frac{d p}{d L} = \frac{f}{D} \rho \frac{V^2}{2} \ldots\ldots\ldots\ldots\ldots (16)$$

and,

$$-\frac{\Delta p}{\gamma} = -\Delta h = \frac{f L}{D} \frac{V^2}{2 g} \ldots\ldots\ldots\ldots\ldots (17)$$

If the pipe slopes, a further change in pressure intensity from one section to the next is essential to balance the longitudinal component of the fluid weight between the two normal sections. It will be clear from Fig. 1,

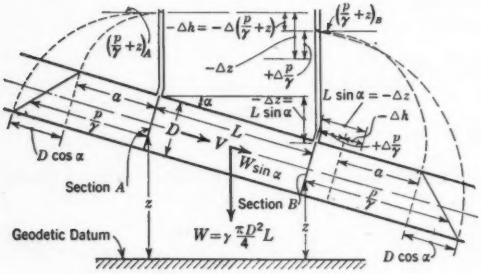

FIG. 1.

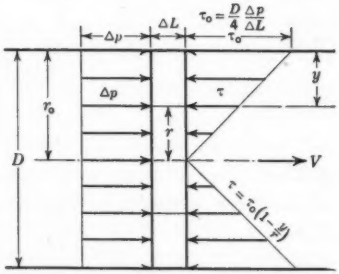

FIG. 2.

however, that this occasions no change in the foregoing relationship between wall shear and that portion of the pressure drop with which it is in equilibrium; hence the assumption of a horizontal pipe merely eliminates two equal and opposite force components from the equation. That is, it makes no difference how steeply the pipe slopes, or whether the flow is the result of pumping or of direct feed from a reservoir; the state of flow at a given section is completely defined once the Reynolds number is known, and is independent of the magnitude of the static pressure load upon the system.

General use of Equation (17) is to be avoided, therefore, unless one is completely familiar with its significance. Involving as it does the specific weight of the fluid (note the open manometer columns in Fig. 1, in a system presumed to be entirely confined by pipe walls), this again brings into the picture the force of gravity, which has just been shown to have no influence upon the flow itself. Unfortunately, the slope of the hydraulic gradient is a function not only of the Reynolds number, but of the Froude number as well—and only if both numbers are the same, respectively, will the grade lines of two different systems be geometrically similar. Since the Froude number refers to flow with a free surface, and actually has no relationship whatever with the inner mechanism of confined flow, no further mention of specific weight will be made in this paper, and Equations (13) and (16) will be considered the basic expressions for pipe resistance.

In developing Equation (15) the assumption is made that the pressure intensity varies statically across any normal section of the pipe. That this must be true will be seen from the fact that neither shear nor acceleration can occur as a continuous process in a direction normal to the pipe axis, for the general movement of the fluid must be in the longitudinal direction. Hence, the only variation in pressure intensity along a pipe diameter must be that necessary to compensate for the change in elevation from one point to the next; that is, the pressure distribution must be static. Thus, the difference in pressure per unit length for any cylinder of fluid within the pipe (see Fig. 2) must vary directly with the square of the radius, r, of the cylinder. Since this difference in pressure must be exactly counterbalanced by

the shear along the outer surface of the fluid cylinder, Equation (14) may be written in terms of any radius, r:

$$(p_1 - p_2)\,\pi\,r^2 = 2\,\pi\,r\,L\,\tau \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

Equation (15) will then become:

$$-\frac{dp}{dL} = \frac{4\,\tau_0}{D} = \frac{2\,\tau}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Expressing the pipe diameter, D, as twice the radius, r_0,

$$\tau = \tau_0 \frac{r}{r_0} = \tau_0 \left(\frac{1 - y}{r_0}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

In other words, the intensity of shear between adjacent layers of fluid co-axial with the pipe wall will vary directly with the distance from the axis of the pipe, as plotted in Fig. 2. It is upon this very general relationship, which holds for laminar and turbulent flow in either smooth or rough pipes, that further analysis of pipe resistance is based.

If the flow is laminar (that is, if the shearing stress is purely viscous, according to the relationship, $\tau = \mu \frac{dv}{dy}$), means are now at hand for determining the velocity distribution in terms of the wall shear. Introducing the relationship,

$$\tau = \mu \frac{dv}{dy} = -\mu \frac{dv}{dr} \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

in Equation (20),

$$dv = -\frac{\tau_0}{\mu\,r_0}\,r\,dr \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

and integrating,

$$v = -\frac{1}{2}\frac{\tau_0}{\mu\,r_0}\,r^2 + C \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

The value of the constant of integration may be found through use of the fact that when $r = r_0$, $v = 0$; that is, the velocity at the pipe wall is zero. Under these conditions,

$$C = \frac{1}{2}\frac{\tau_0\,r_0}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

and the equation for velocity distribution then becomes:

$$v = \frac{1}{2}\frac{\tau_0}{\mu}\left(\frac{r_0^2 - r^2}{r_0}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

Inspection will show that the distribution curve is parabolic, attaining its maximum value at the center, where,

$$v_{max} = \frac{1}{2}\frac{\tau_0 r_0}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

Since the volume of a paraboloid of revolution is one-half that of a circumscribing cylinder, the mean velocity must be one-half the maximum:

$$V = \frac{1}{2} v_{max} = \frac{1}{4} \frac{\tau_o r_o}{\mu} \quad \text{.....(27)}$$

and,

$$\tau_o = \frac{8 \mu V}{D} \quad \text{.....(28)}$$

Through substitution in Equation (15),

$$-\frac{dp}{dL} = \frac{32 \mu V}{D^2} \quad \text{.....(29)}$$

Similarly, substitution of Equation (28) in Equation (13) yields the fundamental dimensionless relationship for laminar flow,

$$f = \frac{64}{R} \quad \text{.....(30)}$$

in which case the exponent, n, must have a value of unity. The problem of laminar flow is thus fully solvable mathematically.

Abundant experimental data exist to vindicate these expressions. For instance, a logarithmic plot of τ_o against V, for a given pipe and a given fluid, results in a single straight line with a slope of unity for the entire laminar range. Once turbulent flow has begun, however, the slope of the line changes (see Fig. 3) from unity to a value varying from $n = 1.75$ for

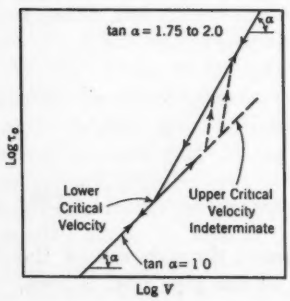

FIG. 3.

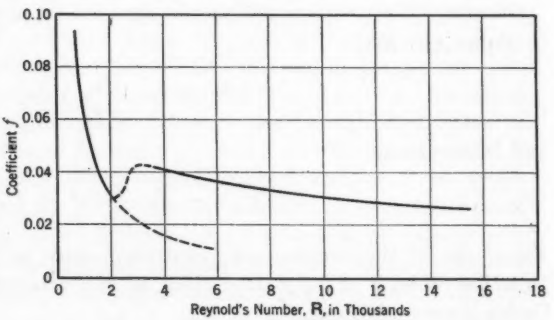

FIG. 4.

extremely smooth pipes to a maximum of $n = 2$ for pipes that are very rough. Aside from this elementary although highly instructive knowledge of the significance of the exponent, n, such a plot is of little aid in drawing general conclusions about the relationships between various systems of flow. The curve is not applicable to more than the one pipe and the one fluid for which it was determined; nor is it of use in other dimensional systems without bothersome revision of scale.

Those engineers who have followed the publications of laboratories in which the metric system is used will readily admit the superiority of dimensionless co-ordinates, if only from a utilitarian point of view. Moreover,

the use of dimensionless parameters often enables the investigator to discover very general principles which otherwise would be obscured in the maze of experimental data plotted in dimensional form. For instance, Equation (16) may be rewritten in dimensionless form through solving for f and combining with Equation (12):

$$f = -2\frac{dp}{dL}\frac{D}{\rho V^2} = \frac{C''}{R^{2-n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

This expression for f as a function of the Reynolds number, another dimensionless quantity, at once suggests the use of these two flow characteristics as ordinates and abscissa for plotting on a single dimensionless graph the variation of the coefficient over the widest possible range of pipe diameter, velocity, viscosity, and density. The form of such a curve for smooth pipes (Fig. 4) is probably already quite familiar to the majority of engineers, for its efficacy in co-ordinating the results of many investigators is undeniable.

Despite the many points in favor of this method, engineers are wont to protest that although such a curve is excellent for very smooth pipes, almost no pipes used in engineering projects are smooth enough to fall within this class. Moreover, the few attempts that have been made to bring commercial rough pipes into such a general relationship have not met with complete success[4]. As a result, the treatment of roughness by engineers has of necessity remained almost purely empirical.

In order to answer such an argument emphatically, it is only necessary to cite the progress that has been made in the present century in this very attempt at generalization. Although the results are quite well known, so far as smooth pipes are concerned, it should prove encouraging to the Engineering Profession to know that progress in the study of rough pipes is continuing successfully at a fully equivalent rate. Research of the present day is emphasizing more and more the general nature of the problem, in an attempt to determine universal relationships which will include every type of pipe that is known, under all possible conditions. It is with the present status of this research that the remainder of this paper will deal.

Physical Characteristics of Turbulence

Attention has already been called to the fact that the Reynolds number is a direct indication of the relationship between the inertial qualities of the flow, which tend toward instability, and the viscous qualities, which have a stabilizing influence. Reynolds' original visual experiments to determine the upper critical velocity—that point at which laminar motion changes to turbulent motion—were splendid illustrations of the growth of instability. Reynolds' equipment consisted of a large tank of still liquid, from which led a glass tube with well-rounded entrance, at the end of which was a valve to regulate the discharge. By introducing a thin filament of dye at the entrance of the tube during discharge, the condition of the flow could be

[4] "Elemente der technischen Hydromechanik", von R. von Mises, I. Teil, B. G. Teubner, Leipzig und Berlin, 1914; "Die Messung der hydraulischen Rauhigkeit", von L. Hopf, *Zeitschrift für angewandte Mathematik und Mechanik*, Band 3, S. 329, 1923; and "A Study of the Data on the Flow of Fluids in Pipes", by E. Kemler, *Transactions*, A. S. M. E., Hydraulics Div., August 31, 1933.

observed by eye over the entire length of the liquid column. So long as the flow remained laminar, the ribbon of color continued as a straight line without diffusing; but the moment the velocity passed above the critical, the ribbon began to waver and finally to diffuse over the entire cross-section.

At the point of initial wavering of the colored filament Reynolds found a mean value of R ranging from 12 000 to 14 000. It is very important to note that later experimenters managed to increase this value of the Reynolds number by improving the entrance, eliminating all vibration, and allowing the liquid in the tank to stand for a much longer period—for several days, in fact. With Reynolds' original equipment, for instance, a value of more than 40 000 has been obtained[5]. The obvious conclusion to be drawn from these experiments is that for perfectly smooth pipes the limiting upper critical point is indeterminate (see Fig. 3), for it depends largely upon the care with which the experiments are conducted. In every case the initial turbulence was a result of some irregularity of flow that was too minute to be discovered beforehand, yet which gradually upset the equilibrium within the fluid.

On the other hand, below a fairly well established point called the lower critical velocity—the point of change from turbulent to laminar flow—the viscous forces are sufficiently large to quell any disturbances that may be present and thus prevent general instability of the flow. Above this lower critical point, which has been almost universally accepted to be about $R = 2\,000$, the slightest disturbance in the oncoming flow is enough to cause a rapid growth of that irregularity of motion called turbulence.

The word "turbulence" has its distinct meaning to those scientists who are now responsible for its analytical investigation, yet it has various independent and often conflicting connotations in the engineering world. To the practical hydraulician engaged in irrigation and power development, for instance, the word instantly suggests the picture of large swirls and eddies which have no relation whatever in his mind with the tiny irregularities of flow in Reynolds' glass tube. Because of this apparently ambiguous usage of the word, the writer submits the following brief picturization of the turbulent phenomenon.

On the small scale of Reynolds' experiments, or on the large scale of the hydraulician's practical experience, fully developed turbulence is basically the same. Superimposed upon the translatory movement of the fluid is a complex internal agitation of the fluid mass, differing from the kinetic motion of gaseous molecules, however, in that finite portions of the fluid rather than infinitesimal particles are involved in the mixing process. Although the action is apparently fully disordered, a statistical treatment of the conditions at any point in a "steady" flow will yield a temporal mean value of the velocity which is invariable so long as the general conditions of flow remain unchanged.

In addition to this temporal mean, the statistical method introduces the conception of instantaneous velocity fluctuations at a given point, as components to be added to the temporal mean velocity measured at the point in

[5] "On the Change from Steady to Turbulent Motion of Liquids," by V. W. Ekman, *Arkiv für matematik, astronomi och fysik*, 6, Nr. 12, 1911.

question; that is, at any instant there may be superimposed upon the translatory movement of the fluid past that point either positive or negative velocity components in each of the three co-ordinate directions. Thus, there is a variable movement of small fluid masses in a direction normal to the general flow, these masses carrying into other regions increments of momentum or energy from the region from which they came—as a result, the turbulent mixing process involves a constant transfer of energy among the various portions of the flow.

Although this statistical interpretation of the turbulence phenomenon is of advantage in the mathematical treatment of energy distribution, only to the mathematicians can it offer a sufficiently complete picture of the internal fluid mechanism. In order to broaden the viewpoint, recourse is often taken to the concept of vortex motion as the root of fluid turbulence. The vortex —briefly, a fluid mass rotating about a central axis and inducing a velocity field in the surrounding fluid medium—is often the source of turbulence, and its rotational motion is fairly typical of the swirls and eddies in a moving stream[6]; yet while turbulent agitation might be thought of as an intricate system of vortices, large and small—some spinning violently and some very slowly, and each influencing the velocity of the neighboring fluid—such a schematic representation gives somewhat too ideal an impression of true turbulence.

Perhaps the best picture is a combination of the two extremes, in which from the statistical representation is taken the notion of continuous—but not abrupt—fluctuation at a given point, and from the vortex conception the visualization of irregular rotational movement of small masses of the fluid in addition to the general movement of the stream. As illustrative of the final picture, imagine dense smoke clouds issuing from a chimney; the motion is obviously changing from instant to instant, but it exhibits on the other hand a very typical sequence of rolling billows of smoke, the general movement of which is by no means the individual helter-skelter of tiny particles of soot; that is, one must not draw from the statistical method the conclusion that each fluid particle describes its own irregular path irrespective of the motion of its neighbors; instead, the behavior of every particle is completely dependent upon that of the others in its immediate vicinity. Free flight and resulting collision between gaseous molecules, furthermore, is not the basic movement in turbulent agitation of fluid masses; nor may the notion of molecular impact, or even of intermittent impact of fluid masses, be considered generally typical of turbulence. On the contrary, the swirling masses are always in complete contact with neighboring masses, and the consequent energy losses are entirely viscous in nature as a result of the variable velocity gradient across any section of the agitated fluid.

A fact commonly ignored at the present time is the existence of considerable energy of turbulence in a moving fluid. The familiar Bernoulli theorem includes only a single kinetic term, velocity head, which is computed from the mean velocity of flow over the entire cross-section. Needless to say, the

[6] Refer to the photographs at the end of the Engineering Societies Monograph, "Applied Hydro- and Aeromechanics", by Prandtl-Tietjens, McGraw-Hill, 1934.

existence of velocity fluctuations indicates further kinetic energy in the fluid, which, unlike the components of fluctuation, has by no means a temporal mean value of zero. This energy of turbulence is perhaps best illustrated by referring once again to the notion of vortices superimposed upon the general flow; however much the individual vortices may be distorted in the actual flow pattern, it is apparent that each single vortex represents a certain amount of rotational energy in addition to its kinetic energy of translation.

The hydraulic engineer has long been familiar with such turbulent energy, under perhaps a different name. The so-called "energy loss" in the hydraulic jump, at "stilling" racks, in tumble bays, at pipe valves, and in various other devices of a similar nature, is really a conversion from the energy of flow embodied in the three terms of the Bernoulli theorem into energy of turbulence. That this transformation is not directly into heat is shown by the fact that the resulting turbulence, of however fine a type (that just after a stilling rack, for instance) is still apparent quite some distance down stream. Eventually, of course, viscous action produced by the turbulent mixing process accomplishes the mechanical transformation into heat, but this is a gradual and not an abrupt process; consider, as illustration, the time necessary to still the water in Reynolds' experiments, or the distance down stream the spiral effect of elbows makes itself known. To be sure, such energy of turbulence is lost to the engineer, for it can never be regained in useful form; but as turbulent energy it can still play insidious havoc as it proceeds down the pipe or channel. C. Camichel[7], for instance, produced artificial turbulence in normal laminar motion by placing a screen in the path of the flow. The resistance then followed the turbulent law until the stabilizing viscous action had restored the flow to its original character some distance down stream. Thus, the presence of abnormal turbulence can change completely the general characteristics of flow.

The causes of turbulence are several in number, a clear understanding of which would depend upon a finer perception of the physics of fluid motion than can be presented in this paper[8]. Mention has already been made of the steady growth of any small irregularity of the flow at Reynolds numbers above the critical, resulting eventually in fully developed turbulence. Furthermore, under certain conditions, roughness of the boundary surface may produce a turbulent "wake" in much the same fashion as will the motion through the air of a vehicle that has not been stream-lined; thus, the action of wall roughness may serve to augment the normal turbulence of the flow. In the latter case the action at the wall will be in effect a constant transformation of potential energy into turbulent energy, which produces as a result an apparent shearing stress between wall and fluid.

Many investigations to determine the "friction factor" for different types of pipe have been of little direct value to the Engineering Profession because insufficient effort was made to distinguish between actual wall resistance and induced resistance caused by turbulence resulting from valves, pipe-joints, or by too great proximity of the measured section to entrance or exit.

[7] "Notice sur les Travaux Scientifiques", Toulouse, 1929.

[8] See, for instance, "Entstehung der Turbulenz", von L. Prandtl, *Zeitschrift für angewandte Mathematik und Mechanik*, Band 11, 1931.

Although the total loss in commercial piping is of direct interest to the designing engineer, it is futile to attempt an analysis of each separate influence if these influences cannot be isolated one at a time. Hence all mention of normal pipe turbulence in this paper refers only to that fully developed turbulent state which will result from the interaction between the fluid itself and the walls of a uniform pipe. It is possible experimentally to make joints of such a nature that they will have no appreciable effect upon the flow, and it has been found that a distance of about 50 pipe diameters from a symmetrical entrance (short elbows or valves may cause so great a disturbance as to require a longer approach section) will be sufficient to permit the full development of normal turbulence.

Resistance as a Function of the Reynolds Number

According to the principle of dynamic similarity, which has had sufficient experimental justification to warrant the complete confidence of both the scientist and the engineer, any single Reynolds number denotes a very definite state of turbulence. Although the flow picture may change from instant to instant because of the complexity of the vortex pattern superimposed upon the general translatory movement, a dimensionless temporal average of velocity conditions at any given section of flow will yield identical results for a given Reynolds number, regardless of fluid properties and pipe diameter. At the same time, it must be remembered that dynamic similarity is dependent upon geometrical similarity, which signifies that the roughness must play a definite part in similitude. Despite the prevalent custom of distinguishing between smooth and rough pipes, it is obvious that roughness is purely a relative characteristic, for even the smoothest surface is rough under the microscope. Recent experimental studies have very clearly demonstrated this fact.

As a general example for the following discussion, assume a pipe that is relatively smooth, remembering that its walls still possess finite surface irregularities. As a given fluid, whether it is a liquid or a gas, flows at a very low Reynolds number through this pipe, the velocity will vary parabolically from zero at the wall to a maximum at the center of the pipe. The fact that the wall velocity is zero is purely a viscous—not an adhesive—effect; for mercury, which does not "wet" the wall, will have just the same flow characteristics as air or water[9].

As the discharge through this pipe is increased, the velocity distribution will remain parabolic, varying only in scale; the shearing stress at the wall will increase in direct proportion to the average velocity (one-half the maximum velocity), as will also the pressure gradient in the direction of flow. Hence, for the given fluid and the given pipe, the frictional factor, f, will be in inverse ratio to the average velocity. Generally speaking, for any fluid and any pipe, so long as the flow is laminar, this friction factor is equal to $f = \frac{64}{R}$. This condition holds until the critical zone is reached (where the

[9] "Unsere Kenntnis der Zähigkeit von Quecksilber", von S. Erk, *Zeitschrift für Physik*, Bd. 47, 1928.

Reynolds number is approximately 2 000), at which point the viscous forces are no longer sufficiently great to counterbalance completely the inertial tendency toward instability. Since these viscous stresses are greatest near the walls of the pipe where the velocity gradient attains its highest value, and lowest in the central regions where the velocity gradient approaches zero, it is reasonable to presume that the first signs of instability will appear at some distance from the wall; this presumption may be vindicated experimentally. That wall roughness (when not excessive) is not the cause of the initial turbulence, will be seen from the following discussion of the boundary layer.

Within the critical zone (from $\mathbf{R} = 2\,000$ to about $\mathbf{R} = 3\,000$), under normal conditions a state of complete equilibrium is difficult to produce. The initial turbulence is so closely dependent upon slight disturbances within the pipe (sharp entrance, elbows, a valve, etc.), that it is difficult to distinguish a definite transition curve among the numerous plots of data that exist. Generalized transition curves (after Schiller) are shown in Fig. 5

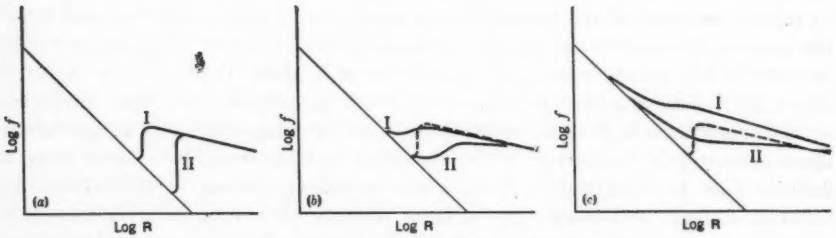

FIG. 5.—GENERALIZED TRANSITION CURVES.

in which the initial disturbance is extremely far away (Fig. 5(*a*)), a short distance away (Fig. 5(*b*)), and, finally, very close to the measured section (Fig. 5(*c*)). Curves marked I represent a large disturbance and II, a small disturbance. Obviously, the smaller and more remote the existing disturbances, the higher the value of **R** at which the change takes place.

However, once the critical zone has been passed, the curve of f against **R** assumes a very definite trend. The earliest measurements of lasting importance are those by Augustus V. Saph, Assoc. M. Am. Soc. C. E., and Ernest H. Schoder, M. Am. Soc. C. E.[10]—the true value of which was realized only after analysis by H. Blasius[11], When plotted by the latter in the now customary form of log f against log **R**, all results in the turbulent region for the relatively smooth (drawn brass) pipes of different diameters lay very close to a straight line with the slope, $-\frac{1}{4}$ (see Fig. 6). Hence, f may be written as inversely proportional to the fourth root of **R**, or, as determined by Blasius:

$$f = \frac{C''}{\mathbf{R}^{(2-n)}} = \frac{0.3164}{\mathbf{R}^{\frac{1}{4}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

[10] *Transactions*, Am. Soc. C. E., Vol. LI (1903), p. 253.

[11] "Das Aehnlichkeitsgesetz bei Reibungsvorgängen in Flüssigkeiten", *Forschungsarbeiten* auf dem Gebiete des Ingenieurwesens, 131, 1913.

This relationship thus corresponds to the exponential form—for a given pipe and a given fluid—of $\tau_0 \propto V^{1.75}$. The logical conclusion to be drawn from this apparently simple result was that all very smooth pipes have an identical

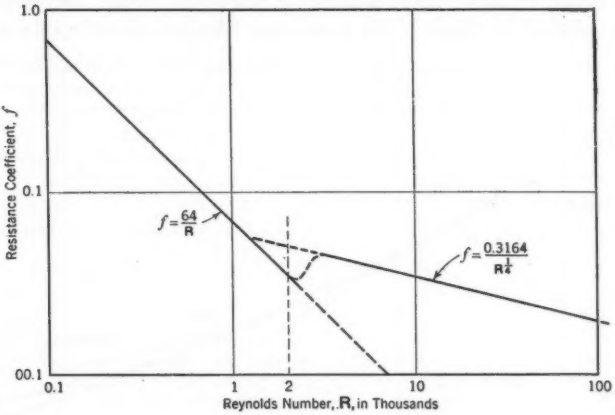

FIG. 6.

effect upon fluid motion, regardless of the degree of turbulence, and that such flow may always be expressed as an exponential function in which $n = 1.75$. (It is to be noted that Reynolds had previously determined this value as 1.732, within surprisingly close range of that of Blasius.)

Nevertheless, research by Messrs. T. E. Stanton and J. R. Pannell[12] soon exceeded the range covered by the curve of Blasius (up to about $\mathbf{R} = 105\,000$) by attaining a maximum Reynolds number greater than 400 000, using both air and water. Based upon these results, C. H. Lees[13] developed the following formula, which differs from that of Blasius in the upper range:

$$f = 0.00714 + \frac{0.6104}{\mathbf{R}^{0.35}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33)$$

About fifteen years later, R. Hermann, after reaching a Reynolds number of nearly 2 000 000, in which range both the Blasius and the Lees formulas were found to deviate from experimental results, proposed the more accurate empirical relationship[14]:

$$f = 0.0054 + \frac{0.396}{\mathbf{R}^{0.3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

This limit has since been surpassed by J. Nikuradse[15], who attained a value of $\mathbf{R} = 3\,240\,000$. These investigations have proved conclusively that the magnitude of n is not constant even for very smooth pipes, but that beyond the Blasius range ($\mathbf{R} = 105\,000$) it increases slowly with increasing

[12] "Similarity of Motion in Relation to the Surface Friction of Fluids", *Philosophical Transactions*, Royal Soc. of London, (A) 214, 1914.

[13] *Proceedings*, Royal Soc. of London, (A) 91, 46, 1915.

[14] See "Rohrwiderstand bei hohen Reynoldsschen Zahlen", von L. Schiller, Aachener Vorträge, S. 69, Springer, Berlin, 1930.

[15] "Gesetzmässigkeiten der turbulenten Strömung in glatten Rohren", V.D.I., *Forschungsheft* 356, 1932.

values of **R**. Thus, flow through an absolutely smooth pipe—which may never be realized—would probably become less and less dependent upon the Reynolds number as the latter increased, attaining complete independence ($n = 2$), however, only when **R** became infinitely great. Measurements by Nikuradse, plotted in Fig. 7, lie upon the Blasius curve up to a Reynolds

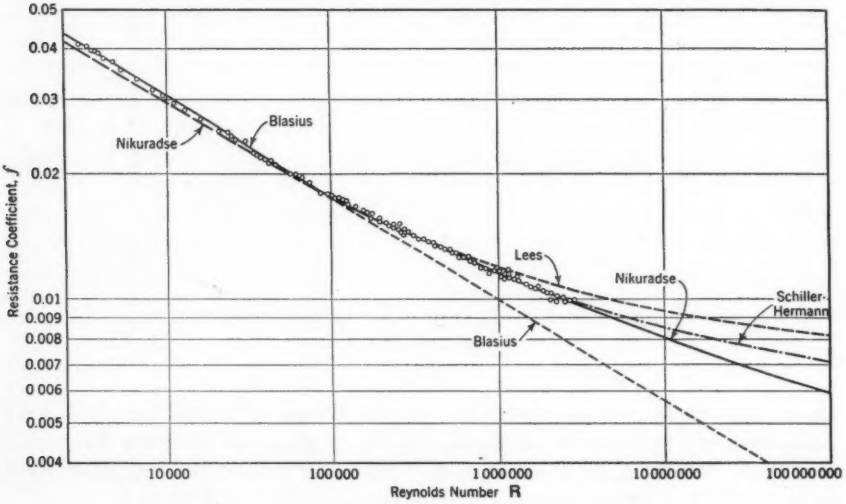

FIG. 7.

number of about 100 000, beyond which point they follow well the curve of Schiller-Hermann. For reasons which will be explained subsequently, Nikuradse has proposed the relationship,

$$f = 0.0032 + \frac{0.221}{R^{0.237}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

which may be expected to apply to Reynolds numbers as high as 10 000 000. For purposes of comparison, the extrapolated curves of Blasius, Lees, Schiller-Hermann, and Nikuradse are all shown in Fig. 7.

Eq. 28

Whereas in laminar flow the resistance was seen to be directly proportional to the average velocity, the turbulent state entails far greater resistance than such a function would provide, since the stress at the walls is now proportional to the average velocity raised to a power varying between 1.75 and 2. It is at once obvious that a redistribution of energy must occur within the flow to cause this change in resistance, since a complete parabolic curve of velocity will no longer provide a sufficiently high value of τ_0. Attention has already been called to the fact that turbulence within the moving body of fluid, schematically illustrated by innumerable vortices superimposed upon the translatory movement, causes a continuous transfer of fluid matter between neighboring layers. Thus, fluid masses from the central regions, possessing a higher velocity in the direction of flow, are whirled into regions in which the velocity is lower, and *vice versa*. In other words, there is a

steady passage of energy from the inner regions of flow in the direction of the pipe walls, resulting in a more uniform velocity distribution over the central portion of the pipe. Hence, the temporal mean velocity curve in turbulent flow, as compared with that of laminar flow, is much flatter at the pipe axis, and accordingly much steeper near the walls (see Fig. 8(*a*)). Obviously, the greater the degree of turbulent mixing (that is, the higher the Reynolds number) the flatter the curve will become.

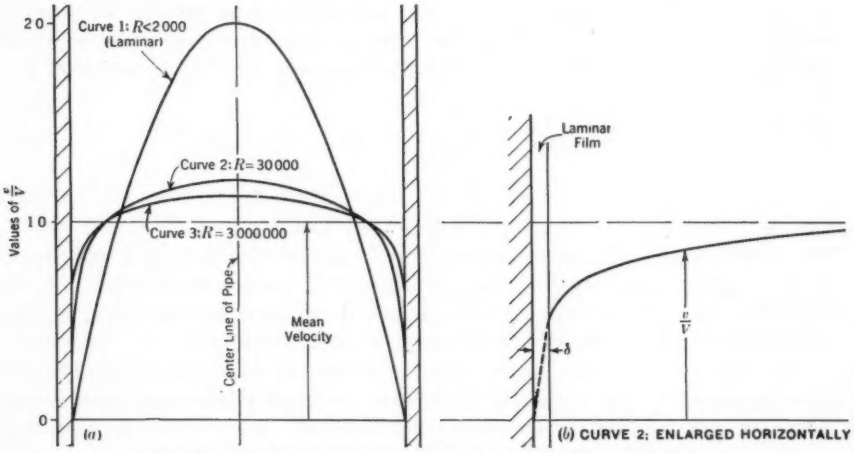

FIG. 8.—VELOCITY DISTRIBUTION IN A VERY SMOOTH PIPE.

Careful measurements of the velocity distribution have long seemed to indicate that the velocity at the wall is by no means zero, as in laminar flow. However, continued improvements in the design of Pitot tubes, enabling a closer approach to the wall, have lowered repeatedly the apparent wall velocity. Despite this indication of a finite velocity at the boundary, analytical investigators have always claimed that the velocity curve must pass through zero at a smooth wall. Based upon this hypothesis, Prandtl developed the theory of the laminar boundary film to account for the high shearing stress at the wall, at which point there is obviously no further turbulent interchange of momentum.

This theory of the boundary layer assumes not only that the wall velocity is zero, but also that the layer of fluid next to the wall contains only a laminar movement of the fluid particles. Since the velocity gradient here will be high —that is, of such magnitude as to provide the shearing stress required by the existing conditions of flow ($\tau_0 = \mu \frac{dv}{dy}$)—the error in assuming a linear velocity distribution, rather than a parabolic distribution, will be quite negligible (see Fig. 8(*b*)). It is evident that the boundary layer must become increasingly thin with ascending values of R, so that when the Reynolds number is high a rough idea of its thickness may be found from the following simple approximation. The shearing stress at the wall for turbulent flow

may be written: $\tau_o = f \rho \frac{V^2}{8}$ (see Equation (13)). This stress must be equivalent to the viscous stress at the wall caused by the velocity gradient, which is assumed to be linear over the small normal distance, δ, denoting the thickness of the laminar film. As an arbitrary value for the velocity at the edge of this film may be taken the apparent wall velocity as noted in measurements. Needless to say, the magnitude of this value is open to debate, since the ratio between the apparent wall velocity, v_w, and the average velocity of flow, V, increases with increasing Reynolds number and decreases with improved methods of measurement. Expressing the foregoing equality in terms of this arbitrary ratio: $\tau_o = f \rho \frac{V^2}{8} = \mu \frac{dv}{dy} = \mu \frac{v_w}{\delta} = \mu \frac{v_w}{V} \frac{V}{\delta}$, and,

$$\delta = 8 \frac{v_w}{V} \frac{\mu}{f V \rho} = 8 \frac{v_w}{V} \frac{D}{f \mathbf{R}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (36)$$

For purposes of illustration, assuming the ratio of apparent wall velocity to mean velocity as about 0.5, the boundary film in a smooth pipe 1 ft in diameter, carrying water at an average velocity of 10 ft per sec at a temperature of 60° F ($\mathbf{R} = 840\,000$) would then be about 0.0004 ft.

In order to verify this theory, Stanton conducted a very painstaking series of experiments[16] with a special Pitot tube of rectangular cross-section, one side of which was formed by the pipe wall itself. Since this side remained stationary, movement of the outer side of the tube away from the pipe wall simply increased the size of the opening and changed the position of its center. The tube was rated first in laminar flow, in which the velocity distribution was definitely known, these corrections then being applied to measurements in the neighborhood of the wall under turbulent conditions. Nikuradse has found similar values through use of Pitot tubes of several different diameters[15], thus determining the error as a function of tube diameter; by extrapolation, the correct velocity (as would be measured with a tube of zero diameter) was readily available. The results of these experiments served as a striking verification of the theory of the laminar boundary film, as have also more recent tests conducted in wind tunnels of large diameter.

It is erroneous, of course, to consider the laminar film as adjacent to the turbulent region, yet sharply divided from it. Actually there is only an arbitrary limit to the boundary layer, for a transition zone must exist between the two regions, in which zone the motion is neither laminar nor completely turbulent. Thus, while within the laminar film the resistance is purely viscous, it is not until well within the turbulent region (where the velocity gradient is low) that the action of viscous shear finally becomes secondary.

It is now possible to draw a valuable comparison between the laminar and the turbulent régimes. In the former case, the laminar boundary film extends to the very center of the pipe; as the critical zone is reached, this laminar portion of the flow decreases rapidly in radial dimension, and then

[16] "On the Conditions at the Boundary of a Fluid in Turbulent Motion", by T. E. Stanton, D. Marshall, and C. N. Bryant, *Proceedings*, Royal Soc. of London, (A) 97, 1920.

continues to diminish gradually with increasing values of the Reynolds number. The thickness, δ, becomes infinitesimal, if the pipe is extremely smooth, only as n approaches the limit 2 (that is, as $R \to \infty$). The previous statement that initial turbulence in relatively smooth pipes does not occur at the walls now becomes clear, since the wall region, except in very rough pipes, is the last stronghold of laminar conditions.

With the realization that a laminar boundary layer actually exists even in turbulent flow, a comprehension of the part roughness plays in influencing turbulent resistance becomes greatly simplified. Roughness may be described in part by comparing with the radius, r_0, the average distance, ϵ, over which the individual projections extend into the flow section from the pipe wall; thus, the term "relative roughness" signifies the ratio of ϵ to the pipe radius, $\frac{\epsilon}{r_0}$. Obviously, this ratio approaches—but never attains—the limit zero in the case of extremely smooth pipes, and on the other hand in rare instances may approach unity for badly corroded surfaces in which the pipe has become almost completely choked through age; the former class comprises very smooth commercial piping (glass, drawn brass, and lead), but the latter case is so extreme as to make its inclusion in such a paper quite reasonless. In other words, normal relative roughness of pipes can be assumed to vary from nearly zero to perhaps one-fifteenth.

With this partial definition of roughness, it now is possible to subdivide the turbulent region into two distinct zones, separated by a region of transition. It has often been proved that in the laminar régime slight surface roughness has no effect whatever upon either velocity distribution or wall resistance. Similarly, it may be reasoned that so long as the laminar boundary film is of such thickness as to cover all roughness protuberances at the wall, such roughness will continue to have no effect upon the conditions of flow. Thus it is that smooth pipes—despite the fact that they are actually rough to a certain degree—appear to follow a different resistance law from that of commercial rough pipes, for the relative roughness is so slight that the unevennesses are still enveloped in the thin laminar film; yet many pipes, ordinarily classed as smooth, behave as such only for comparatively low Reynolds numbers, since the relative roughness is such that the boundary layer at high Reynolds numbers is not sufficiently thick to cover the roughness particles. Other pipes, in which the relative roughness is high, at no time follow the curve of resistance for smooth pipes beyond the critical zone, because the thickness of the boundary film is never great enough to include all roughness particles in a continuous enveloping layer.

The data plotted by Kemler, previously cited[4], are not in general sufficiently systematic to warrant more than broad statements as to the nature of the absolute action of roughness in the sense already defined. Two conclusions are of great importance, however. One, long since recognized, is that once roughness extends inside the laminar film, the resistance to flow is greatly increased; the second is that roughness may not be defined completely by a ratio of radial dimension to pipe radius, for the plots of certain rough pipes

show complete independence of R, whereas others run approximately parallel to the minimum curve for smooth pipes (Fig. 9).

The conditions which must exist for dynamic similarity, already stated in this paper, include first that of strict geometrical similarity. Thus, in

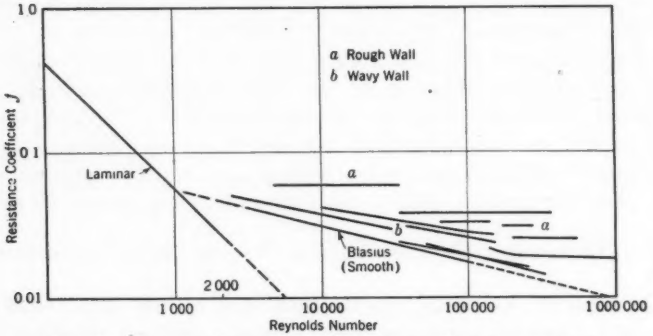

FIG. 9.—TYPICAL RESISTANCE CURVES FOR COMMERCIAL PIPE.

order for flow through any two pipes to be dynamically similar, not only must the Reynolds numbers be identical, but the entire physical character of the inside of one pipe must be similar geometrically to the other; yet this means that not only the relative roughness, but also the general form and spacing of the irregularities themselves, must be the same in each case. Obviously, the relative roughness of rivets inside a penstock may equal the relative roughness of rust tubercles or that of unevennesses in commercial asphalt-lined pipe; yet there is no reason whatever to suppose that the action of all such forms of roughness should be the same. As a matter of fact, a surface may be further defined (refer to Hopf[4]) by distinguishing between waviness and actual jagged roughness; whereas jagged roughness that breaks the continuity of the laminar film results in a constant resistance coefficient for all values of R, the waviness (corrugated, asphalt, or galvanized lining) serves to increase the resistance in such a way that it is still a function of R. Of course, it is not always possible to differentiate between the two forms, as the wall material in some cases may be a combination of both, the resistance curves then lying between the two extremes. Typical plots may be seen in Fig. 9.

As a suggested explanation of the difference in behavior between rough and wavy surfaces, the writer offers the following: When bodies of very angular profile are tested in a wind tunnel, their resistance coefficients are found to be nearly independent of the Reynolds number characterizing the flow; the turbulent wake behind such a body is then determined in size largely by the form and position of the outermost irregularities of the profile. As the angles of the body are softened, approaching the ideal stream-line form, a profile is reached for which the turbulent wake is smaller and not of constant form, actually decreasing in size with increasing Reynolds number, the resistance factor thereby dropping appreciably in magnitude. This phenomenon is explained by the fact that an increase in velocity decreases the thick-

ness of the laminar film surrounding the front of the body to such an extent that the point of separation (now a function of velocity distribution along the softened curves rather than of abrupt angularity) moves toward the rear of the body. In the completely stream-lined profile, separation and the formation of a wake are obviated. Whereas the body of irregular profile corresponds to the projections on a very rough wall and the stream-line profile typifies the smooth wall, there may be irregularities in pipe surface between the two extremes which exhibit much the same variation in turbulent "wake" as the transition profile in the foregoing illustration.

What is considered by several authorities to be a more scientific viewpoint than the notion of relative thickness of laminar film and roughness particles is the following: Every roughness protuberance may be considered to act as an individual body, to which a lower critical Reynolds number may be assigned in much the same way as is done for the plate or sphere; this number is composed of the dimension, ϵ, the kinematic viscosity of the fluid, and the velocity existing at the outermost part of the roughness projection. The latter may be computed on the basis of a linear velocity gradient in the boundary layer, with a velocity of zero at the wall itself. Thus, when the critical number for the projection in question is exceeded, separation occurs, with a resulting turbulent wake. Although it is doubtful whether roughness particles may behave in such an independent fashion unless their spacing is relatively great, several investigators have been able to compute, with fair approximation, the point at which roughness projections of known magnitude will begin to produce wall turbulence.

A great advantage of this viewpoint is the clarity it lends to the action of roughness in the laminar régime. Although the fact has already been mentioned that slight roughness has no influence in purely laminar flow, it is still possible that the local critical velocity for projections of large magnitude may be reached shortly before the critical value of $\mathbf{R}$ for the entire pipe. Although such individual disturbances may be dampened out before affecting the general flow, it is obvious that roughness of large magnitude may provide the initial disturbance leading to fully developed turbulence in the critical zone.

It has been stated that the resistance in the case of either roughness or waviness is a function not only of relative roughness, but also of the form and spacing of the individual particles. Not only is this complex general function still to be determined, but it is also difficult to classify the wide variety of commercial pipes accurately without further knowledge of the function itself. Such knowledge cannot come from the plentiful existing data on commercial pipes, because of the lack of systematic investigation already mentioned. However, a commendable start has been made by Nikuradse[17] and others in the investigation of controlled, artificial roughness. By means of uniform sand grains attached to the pipe wall with a thin coat of varnish, Nikuradse studied not only the similarity of flow with different pipe diameters and a constant relative roughness, but also the variation of f with roughness

[17] Strömungsgesetze in rauhen Rohren", V. D. I., *Forschungsheft* 361, 1933.

of six different relative magnitudes, over a wide range of the Reynolds number. So systematic was his investigation that the flow behavior in all the different zones discussed previously was clearly demonstrated. Reference to the plot (Fig. 10) taken from Nikuradse's experiments will show distinctly

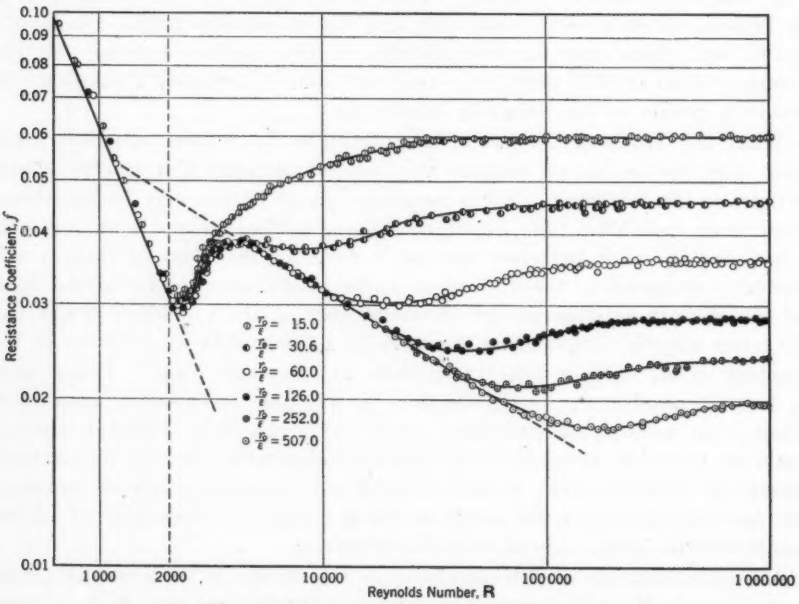

FIG. 10.—MEASUREMENTS ON ARTIFICIALLY ROUGHENED PIPE.

the regions in which the roughness particles were enveloped by the laminar film, the flow then being similar to that through a very smooth pipe. Just as distinct is the transition region, wherein the roughness particles begin to disrupt the boundary layer. Finally, Fig. 10 shows the zone in which the continuous laminar film has disappeared entirely, the friction coefficient then being dependent upon the relative roughness alone. For the last of the three zones Nikuradse has found the relationship:

$$f = \frac{1}{\left(2 \log_{10} \frac{r_0}{\epsilon} + 1.74\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (37)$$

It is to be noted that Equation (37) may be expected to apply generally only in the event that the roughness in question is geometrically similar to that of the sand coatings in Nikuradse's investigation. Any departure in the form of roughness will most certainly require different constants in this relationship, and in some cases (waviness) even a different form of equation.

Although, to a certain extent, Fig. 10 shows the systematic variation of the coefficient with a change in relative roughness, Nikuradse has indicated their very close relationship by plotting them in such a way that all points

in the third region (where f is independent of R) are superposed. As Equation (37) may be changed to read,

$$\frac{1}{\sqrt{f}} - 2\log_{10}\frac{r_0}{\epsilon} = 1.74 \quad \ldots\ldots\ldots\ldots\ldots\ldots (38)$$

which is simply a statement of the constant relationship between f and $\frac{\epsilon}{r_0}$ in this range, the use of the left-hand side of this equation as ordinates for all three regions would result in a series of points which might be expected to coincide only in the region of constant f. As abscissa was selected the quantity, $\sqrt{\frac{\tau_0}{\rho}}\ \frac{\epsilon}{\nu}$, which indicates as a dimensionless number of the Reynolds' type the flow characteristics in the vicinity of the wall; that is, $\sqrt{\frac{\tau_0}{\rho}}$ has the dimension of a velocity (see Equation (13)) while ϵ is the appropriate length parameter.

A glance at this plot as reproduced in Fig. 11 will indicate at once that within the range of experimental error the resistance coefficient will lie upon

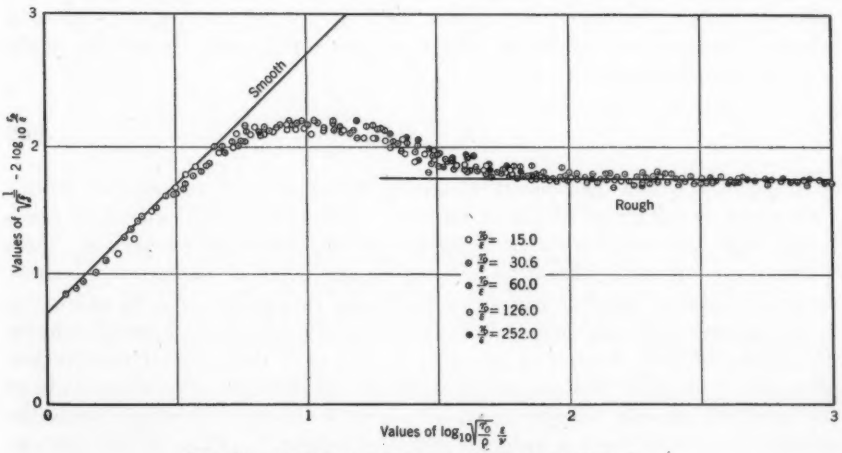

FIG. 11.—COMPOSITE PLOT OF CURVES IN FIG. 8.

a single dimensionless curve, regardless of relative roughness, so long as the roughness is of a similar nature. It is with the investigation of other types of roughness in this systematic fashion that laboratories are now engaged, with the ultimate goal of determining the proper functional relationship for major types of commercial pipe.

VELOCITY DISTRIBUTION AND THE MIXING LENGTH

Notwithstanding the fact that numerous attempts have been made during the past hundred years to formularize the velocity distribution in turbulent flow, very few expressions have been derived which apply correctly to much

more than the limited range of Reynolds numbers for which they were developed. Moreover, nearly all such empirical attempts have included only the central portions of the flow, disregarding entirely the region in the immediate neighborhood of the wall. By far the most scientific approach to the problem of energy distribution in turbulent flow is that proceeding from a general relationship developed by J. Boussinesq[18]; since the shearing stress at any point in purely laminar motion may be expressed as the product of the fluid viscosity and the velocity gradient (Equation (21)), Boussinesq reasoned that the intensity of shear in purely turbulent motion may be written in a closely parallel form,

$$\tau = \eta \frac{dv}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

In Equation (39), however, the factor, η, unlike μ, is not a constant for a given fluid, for it expresses the dynamic quality of the turbulent process at a given point; η will thus depend not only upon the fluid density, but upon the general state of flow and upon the point in the flow which is being considered. The gradient, $\frac{dv}{dy}$, as shown by Boussinesq, will now be that of the temporal mean velocity. Inasmuch as viscous shear is always present, regardless of whether the flow is laminar or turbulent, the total shear must then be the sum of these two quantities:

$$\tau = \mu \frac{dv}{dy} + \eta \frac{dv}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

Osborne Reynolds has shown that the last term may be expressed as a function of the density and of the momentary fluctuations in velocity at a given point, since the latter are a true measure of the degree of turbulence. Such fluctuations will be apparent at any stationary point of observation because of the transit of whirling masses of fluid past this point; that is, there will be superposed upon the general velocity of translation the tangential velocity component of each portion of the eddy as it passes the point of observation. Obviously, this will result in a rapid change in direction and magnitude of the resultant velocity, followed immediately by a different variation during the transit of another eddy of different size and intensity. Thus, at any instant there must be added, vectorially, to the temporal mean velocity, v, at the point in question the velocity of fluctuation, v', which varies constantly in magnitude and direction, having two essential components, v'_x, parallel to, and v'_y normal to, the pipe wall; the third component, normal to the other two, is of no consequence in this development. It is evident that both v'_x and v'_y must yield a temporal mean value of zero; the temporal mean of the product of the two essential components, $\overline{v'_x v'_y}$, however, will always have a finite value, except at the pipe wall and pipe axis, because of a certain correlation which exists between the two components in the normal turbulent pattern.

[18] "Essai sur la théorie des eaux courantes", Mémoires présentés par divers savants a l'Académie des Sciences de l'Institut de France, Tome 23, 1877.

As a result of these velocity fluctuations, at any instant through a small increment of area, Δa, normal to the pipe radius, there will be a rate of flow equal in magnitude to $\Delta a\, v'_y$. A positive magnitude of this value denotes an instantaneous rate of flow toward the pipe axis—that is, into a region where the velocity is higher—and *vice versa*. Because of this movement into a region of different velocity, at that instant there is being transported into the neighboring region of flow either a positive or a negative increment of momentum. The component of this increment in the direction of average flow is proportional to the density, ρ, and to the velocity fluctuation in this direction, v'_x. The rate of passage in the y-direction of this axial component of the increment of momentum is then given by the quantity, $-\Delta a\, v'_y\, \rho\, v'_x$, which has the dimension of a force. The negative sign signifies that fluid moving in the negative y-direction (toward the wall) into a region of lower velocity exerts a positive force. Hence, the force per unit of area, or the apparent shearing stress due to turbulence, must have a temporal mean value of $-\rho\, \overline{v'_x\, v'_y}$, and,

$$\tau = \mu \frac{dv}{dy} - \rho\, \overline{v'_x\, v'_y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41)$$

Since viscous shear within a given fluid must depend entirely upon the velocity gradient, it will be evident that such viscous action has two quite different functions: Owing to the presence of innumerable eddies, the local velocity gradient at a given section may be quite high across each of these eddies at the instant of transit, the curve of instantaneous velocity distribution then differing considerably from its temporal mean form. The function of this relatively intense local shear is to dissipate the energy of turbulence as rapidly as it forms, for only through viscous action can energy of flow finally be converted into heat. In addition to such local stress within the eddy itself, however, there must be a continuous viscous resistance to general translatory movement through the pipe, the latter resistance being proportional to the gradient of the temporal mean velocity curve. Obviously, the local dissipation of energy of turbulence is of secondary importance in this discussion, however vital the rôle it must play in maintaining the essential equilibrium of the fully developed turbulent state.

It so happens, moreover, that the magnitude of the viscous resistance to the general translatory motion becomes, at high Reynolds numbers, of inappreciable magnitude in comparison with that resulting from turbulent transfer of momentum. This will be seen from Fig. 12, in which are plotted, against dimensionless co-ordinates, data found by Nikuradse for smooth pipes at Reynolds numbers ranging from slightly above the critical to a maximum of more than 3 000 000. Not only does the slope of the velocity curve in the turbulent portion of the flow decrease with increasing Reynolds number, but the total resistance as evidenced by the slope at the wall becomes ever greater as the average viscous shear in the central regions diminishes. This trend is shown in a somewhat different form by Fig. 13, in which is plotted the ratio of the average velocity to the maximum velocity as a function of the

Reynolds number, based upon the results obtained by various investigators for smooth pipes; as is readily apparent from Fig. 13, the higher the values of R, the more nearly equal the average and maximum velocities become.

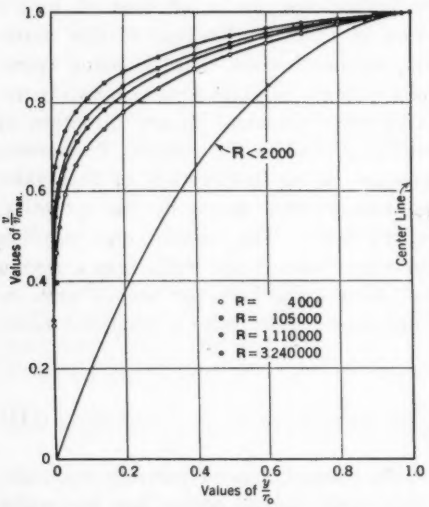

FIG. 12.—VELOCITY CURVES FOR SMOOTH PIPES.

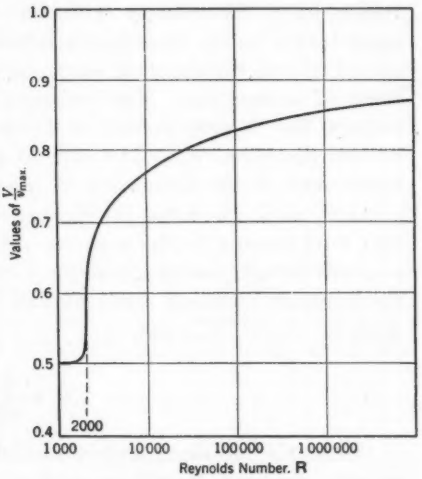

FIG. 13.—RATIO OF MEAN TO MAXIMUM VELOCITY AS A FUNCTION OF R.

Experimental results show that beyond the range of the Blasius equation for smooth pipes the term for average viscous shear may be omitted entirely without introducing noticeable error, the region of predominant viscous influence being restricted entirely to the vicinity of the pipe wall. In the case of rough pipes in the range of constant f, the turbulent action is so excessive that viscous resistance as a function of the mean velocity gradient is completely negligible in all parts of the flow. Thus, the Boussinesq concept of purely turbulent resistance in the central regions of flow is practically realized throughout the realm of high Reynolds numbers for smooth and rough pipes alike.

Before general use can be made of the Reynolds relationship for shear, it is necessary to find a means of expressing the quantity, $\overline{v'_x\, v'_y}$, in terms of known characteristics of flow. Prandtl made the first essential step in this direction by assuming a certain mixing length—analogous to the mean free path in the kinetic theory of gases—to denote the average transverse distance, in any region of the flow, over which a small fluid mass is carried by the turbulent mixing process. It then follows that the difference in average velocity between the beginning and the end of this distance (approximately equal to the product of the mixing length and the velocity gradient, $l\,\frac{dv}{dy}$) must be a measure of the component of fluctuation, v'_x, in the axial direction; furthermore, the existence of correlation between v'_x and v'_y denotes a general

proportionality between these two components; that is, the difference between the original average velocity of the particle and that of the region into which it comes must be proportional to the magnitude of the velocity fluctuations involved in this transverse motion. This proportionality may be written: $l \frac{dv}{dy} \propto v'_x \propto v'_y$, and $\tau \propto \rho\, l^2 \left(\frac{dv}{dy}\right)^2$. By changing this proportionality to an equality, l will absorb the proportionality factor, and will then be completely defined:

$$\tau = \rho\, l^2 \left(\frac{dv}{dy}\right)^2 \qquad (42)$$

At this point it is interesting to note that Boussinesq's coefficient of turbulence, η, through division by the density, ρ, will take on dimensional characteristics similar to those of the kinematic viscosity, ν, the ratio then denoting the kinematic characteristics of the turbulent process in a manner quite independent of the fluid properties. By combining Equations (39) and (42), there results the following significant expression for this kinematic parameter:

$$\frac{\eta}{\rho} = l^2 \frac{dv}{dy} \qquad (43)$$

Since the shearing stress at any point within the flow is directly proportional to the distance from the wall (Equation (20)), Equation (42) then becomes:

$$\tau_0 \left(1 - \frac{y}{r_0}\right) = \rho\, l^2 \left(\frac{dv}{dy}\right)^2 \qquad (44)$$

It is to be noted that Equations (42) and (44) do not include viscous shear; hence, they cannot be expected to apply to those conditions of flow in which viscous resistance is appreciable.

Nikuradse has computed the magnitude of l from measured values of wall resistance and measured curves of velocity gradient for smooth pipes[15] over a great range of R, and for rough pipes[17] with various values of $\frac{\epsilon}{r_0}$ within the region of constant f; these values are plotted in dimensionless form in Fig. 14. In every case the mixing length has a value of zero at the wall (actually at the edge of the boundary layer, for at this point there is no further velocity fluctuation), rising to a maximum at the center. As is to be expected, the computed mixing length for smaller values of R is greater in magnitude than is actually the case, since Equation (44) ignores the viscous influence, which is still appreciable in this range of R. Beyond the limit of $\mathsf{R} = 105\,000$, however, l is practically independent of R, apparently depending only upon the relative distance from the wall. Comparison with the single curve drawn through the points for rough pipes would seem to indicate that this function is invariable, regardless of the magnitude of the relative roughness.

Thus, although viscous shear still plays an important rôle throughout the moving fluid even well past the critical zone, influencing the internal mechanism of motion (and, hence, the velocity distribution and wall resistance) this

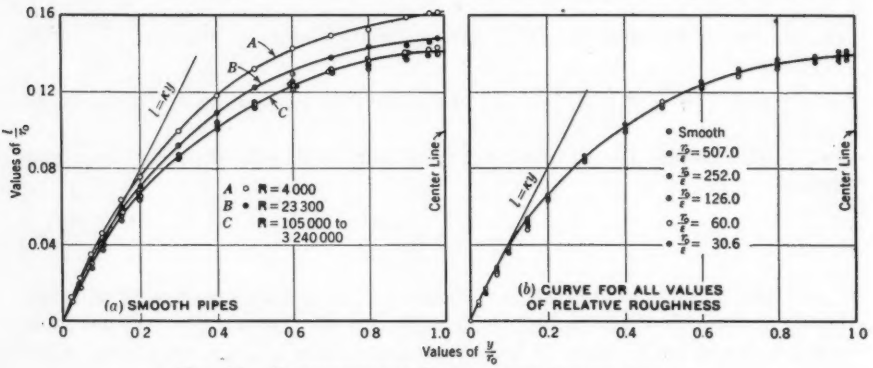

FIG. 14.—DIMENSIONLESS PLOT OF THE MIXING LENGTH.

general influence decreases steadily with rising values of R, until finally a limit is reached beyond which viscous resistance to flow is completely negligible throughout the greater portion of the pipe, being restricted to a relatively thin boundary region. Furthermore, experiments indicate that the variation of the mixing length for smooth pipes beyond this limit is identical with that for rough pipes in a corresponding range of R, showing that the internal mechanism of flow must be quite similar in all such cases.

UNIVERSAL CHARACTERISTICS OF TURBULENT FLOW

Through a statistical analysis of the velocity fluctuations at any point within turbulent movement past a fixed surface, Theodor von Kármán[19], M. Am. Soc. C. E., has concluded that (where viscous resistance to flow is negligible) there must exist a certain correlation between the components, v'_x and v'_y, regardless of the distance from this surface (excluding, of course, the boundary region). Without venturing into a detailed discussion of the theory of correlations, the following reasoning will serve to illustrate this relationship: Particles of fluid moving away from the wall into a region of higher velocity as the result of a positive v'_y-fluctuation will have an average velocity, v, smaller than that of this new region. Since this difference will appear as a negative v'_x-fluctuation at this point, it is obvious that positive values of v'_y are generally associated with negative values of v'_x and *vice versa*. Experiments made in the United States and in Europe with a hot wire anemometer capable of recording rapid fluctuations have verified von Kármán's conclusion that correlation must exist, these investigations indicating a constant correlation factor from the wall region almost to the center of the pipe.

As was shown herein under the heading "Velocity Distribution and the Mixing Length", there exists experimental evidence that the ratio of the mixing length to the pipe radius is a definite function of relative distance from

[19] "Mechanische Aehnlichkeit und Turbulenz", *Göttinger Nachrichten*, Math. Phys. Kl., 1930; see, also, by the same author, "Turbulence and Skin Friction", *Journal of the Aeronautical Sciences*, January, 1934.

the wall, and of no other characteristic of the flow. Hence, there appears to be a certain relationship between the turbulent patterns in any two pipes, regardless of pipe diameter, relative roughness, velocity, and fluid properties. As pointed out by von Kármán, the constant correlation of the components of velocity fluctuation indicates, furthermore, a similarity of the turbulent mechanism over the larger part of the cross-section of any given pipe; that is, not only should corresponding regions in two pipes show similar movement, but flow in any region in the one pipe should be similar to that in any region in the other pipe, the turbulent pattern differing only in time and length scales. (The variation in the length scale, for instance, may be typified by the variation of the mixing length, while the time scale is associated with the ratio, $\frac{dv}{dy}$.) Moreover, because of this similarity of the turbulent pattern, the conditions at any point of any flow may be expressed entirely in terms of their variation with y in the immediate vicinity of the point in question.

von Kármán has thus expressed the mean velocity, v, at a given point in terms of its variation with increments of y through expansion in a series, omitting, however, all terms after the second derivative. Then writing a general stream function in terms of this velocity series and introducing the proper time and length scales, by means of certain principles of classical hydrodynamics he has deduced the following relationship for the mixing length in terms of a universal constant:

$$l = \kappa \frac{\frac{dv}{dy}}{\frac{d^2v}{dy^2}} \qquad \text{.....(45)}$$

The merit of this expression lies in the fact that the value of l may now be written for any point in the turbulent region without regard for distance from the wall or center line of the pipe. It is to be noted that A. Betz[20] has developed an identical equation in a different manner, working in terms of the spacing and rotational speed of an ideal group of vortices superimposed upon the general translatory motion of the fluid. This fact is quite significant in correlating the statistical method with the physical picture of the turbulent mixing process.

A simple schematic digest of the complex mathematical reasoning leading to Equation (45) is given in the following development presented by B. A. Bakhmeteff, M. Am. Soc. C. E., in his lectures on fluid mechanics. The mean velocity, v, at any two points of the flow, expressed in terms of variation with increments of y according to Taylor's theorem, will be as follows:

$$v = \phi\,(y + \Delta y)$$

$$v_{[y_1+\Delta y]} = v_1 + \frac{dv_1}{dy}\,\Delta\, y + \frac{d^2v_1}{d\,y^2}\,\frac{\overline{\Delta y}^2}{2} + \frac{d^3v_1}{dy^3}\,\frac{\overline{\Delta y}^3}{6} +$$

[20] "Die v. Kármânsche Aehnlichkeitsüberlegung für turbulente Vorgänge in physikalischer Auffassung", *Zeitschrift für angewandte Mathematik und Mechanik*, Band 11, 1931.

and,

$$v_{[y_2+\Delta y]} = v_2 + \frac{dv_2}{dy}\,\Delta y + \frac{d^2v_2}{dy^2}\,\frac{\overline{\Delta y}^2}{2} + \frac{d^3v_2}{dy^3}\,\frac{\overline{\Delta y}^3}{6} + \cdots$$

Owing to the similarity of the flow pattern at Points 1 and 2, the following proportionality must exist between corresponding derivatives of the expansion:

$$\frac{\dfrac{dv_1}{dy}}{\dfrac{dv_2}{dy}} \propto \frac{\dfrac{d^2v_1}{dy^2}}{\dfrac{d^2v_2}{dy^2}} \therefore \frac{\dfrac{d^2v_1}{dy^2}}{\dfrac{d^2v_2}{dy^2}} \propto \frac{\dfrac{d^3v_1}{dy^3}}{\dfrac{d^3v_2}{dy^3}} \text{ ; etc.}$$

Omitting all terms of higher order than the second, this proportionality may then be written:

$$\frac{\dfrac{dv_1}{dy}}{\dfrac{d^2v_1}{dy^2}} \propto \frac{\dfrac{dv_2}{dy}}{\dfrac{d^2v_2}{dy^2}}$$

Obviously, each ratio has the dimension of a length, which means only that the magnitude of the ratio varies directly with the length scale of the pattern; that is, the mixing length is equal to a universal constant times the ratio of the first and second derivatives of the velocity with respect to y, as in Equation (45). In the wall region where y is very small, increments of y will be of the same order as y itself; in this region the mixing length will vary directly with y, as may be seen from Fig. 14; thus $l = \kappa y$.

The magnitude of the universal constant has been found experimentally to lie between the limits of 0.36 and slightly more than 0.40, the small discrepancy resulting to some extent from unavoidable experimental error. Variation within such narrow limits is surely sufficient reason to recognize the existence of a universal characteristic of the flow.

Darcy was probably the first to notice the resulting similarity between curves of velocity distribution in turbulent flow. Such similarity is strikingly displayed in the group of curves shown in Fig. 15(a), taken from measurements by Nikuradse. Each of these curves represents a different magnitude of relative roughness, the similarity arising from the fact that the dimensionless ordinates have been reduced to a common denominator, $\sqrt{\frac{\tau_0}{\rho}}$. The physical significance of this method of plotting is as follows: Equation (13) may be rewritten in the form:

$$\sqrt{\frac{\tau_0}{\rho}} = V\sqrt{\frac{f}{8}} \text{ and } \frac{1}{\sqrt{f}} \propto \frac{V}{\sqrt{\frac{\tau_0}{\rho}}}$$

in which the term, $\sqrt{\frac{\tau_0}{\rho}}$, has the dimension of a velocity and is known in fluid mechanics as the "friction velocity". The foregoing proportionality shows that the dimensionless ratio of mean velocity to friction velocity will

vary inversely with the square root of f, regardless of whether the wall resistance is caused by viscous shear in the laminar film of smooth pipes or by the turbulent "drag" along the walls of rough pipes. In the case of smooth pipes this ratio will increase with decreasing values of f; referred to a common friction velocity, this would mean that the central portion of the distribution curve would merely shift in one direction or the other, the velocity gradient at the walls thereby remaining constant. In the case of rough pipes in the range of constant f, this ratio would of necessity be constant for any given relative roughness, but would decrease with increasing values of $\frac{\epsilon}{r_0}$.

Furthermore, the use of the friction velocity to indicate average conditions at once establishes a similar common denominator for all portions of the flow: Mention has already been made of the fact that since the pressure is statically distributed over the pipe cross-section, the intensity of shear must be a linear function of relative distance from the wall (see Fig. 15(*b*)). In flow at high Reynolds numbers, in which viscous resistance in the central portion of the flow is negligible, the shearing stress must be due entirely to the turbulent exchange of momentum—and this turbulent exchange of momentum at the same time determines the velocity distribution in all regions except the immediate vicinity of the wall. Thus, regardless of what the distribution curve may be in the wall region, the ratio of velocity to friction velocity must display a similar variation over the entire central region of

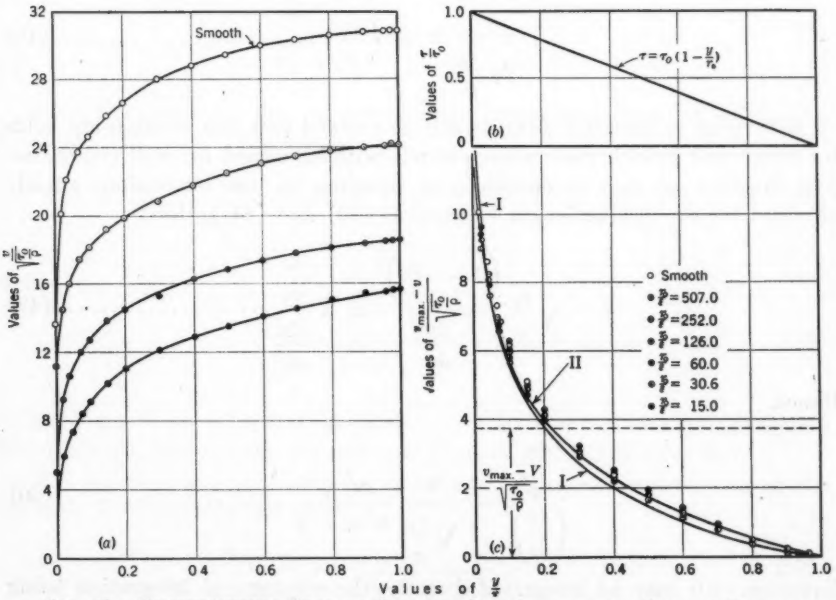

FIG. 15.—DIMENSIONLESS PLOT OF VELOCITY DISTRIBUTION FOR VARIOUS VALUES OF RELATIVE ROUGHNESS.

the flow, for smooth pipes as well as for all values of relative roughness. Hence, if the central portions of all curves in Fig. 15(*a*) are superposed, a single curve will result.

In other words, within the purely turbulent portion of the movement, the difference between the maximum velocity at the center and that at any other point (that is, the "velocity defect"), divided by the friction velocity, is purely a function of the relative distance from the wall. von Kármán was the first to formulate this universal function:

$$\frac{v_{max} - v}{\sqrt{\frac{\tau_o}{\rho}}} = \phi_1\left(\frac{y}{r_o}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46)$$

The conditions near the wall, however, are still dependent upon other influences. The velocity near a smooth surface, where viscous shear predominates, depends upon τ_o, ρ, μ, and y. A completely general expression for this universal relationship must, for dimensional reasons, have the form,

$$\frac{v}{\sqrt{\frac{\tau_o}{\rho}}} = \phi_2\left(\sqrt{\frac{\tau_o}{\rho}}\,\frac{y}{\nu}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47)$$

Similarly, the velocity in the vicinity of a rough wall, where viscous influences do not exist, will depend entirely upon τ_o, ρ, y, and ϵ. The corresponding universal function must then be:

$$\frac{v}{\sqrt{\frac{\tau_o}{\rho}}} = \phi_3\left(\frac{y}{\epsilon}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48)$$

Returning to Equation (44), it will be recalled that this relationship holds for rough and smooth pipes alike, entirely without regard for wall conditions. The function, ϕ_1, may be developed by equating the two expressions already obtained for the mixing length (Equations (44) and (45)), thus:

$$l = \sqrt{\frac{\tau_o}{\rho}}\,\frac{\left(1 - \frac{y}{r_o}\right)^{\frac{1}{2}}}{\frac{dv}{dy}} = \kappa\,\frac{\frac{dv}{dy}}{\frac{d^2v}{dy^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49)$$

Hence,

$$\frac{\frac{d^2v}{dy^2}}{\left(\frac{dv}{dy}\right)^2} = \frac{\kappa}{\sqrt{\frac{\tau_o}{\rho}}}\,\frac{\sqrt{r_o}}{\sqrt{r_o - y}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50)$$

Equation (50) may be integrated directly, the constant of integration being found by assuming an infinite velocity gradient at the wall (this is of no

serious consequence, as the wall conditions play no part in the central regions for which the relationship is valid):

$$\frac{dv}{dy} = \frac{1}{2\kappa} \frac{\sqrt{\frac{\tau_o}{\rho}}}{\sqrt{r_o}} \frac{1}{\sqrt{r_o - y} - \sqrt{r_o}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (51)$$

Integrating once again between the limits, $y = y$ and $y = r_o$, the final relationship is obtained, in accord with Equation (46):

$$\frac{v_{max} - v}{\sqrt{\frac{\tau_o}{\rho}}} = -\frac{1}{\kappa}\left[\log_e\left(1 - \sqrt{1 - \frac{y}{r_o}}\right) + \sqrt{1 - \frac{y}{r_o}}\right] \quad \ldots\ldots (52)$$

A plot of the velocity defect divided by the friction velocity yields a general curve (I in Fig. 15(c)) which fits fairly well all measured values of the velocity within the central regions of flow. In order to take into consideration the outer regions, it is necessary to return to Equations (47) and (48); the only relationship commensurate with these expressions and the similarity of the turbulent pattern must be the following, first derived by von Kármán:

For smooth pipes,

$$\frac{v}{\sqrt{\frac{\tau_o}{\rho}}} = C_1 + \frac{1}{\kappa} \log_e \left(\sqrt{\frac{\tau_o}{\rho}} \frac{y}{\nu}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (53)$$

and, for rough pipes,

$$\frac{v}{\sqrt{\frac{\tau_o}{\rho}}} = C_2 + \frac{1}{\kappa} \log_e \frac{y}{\epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

With Constants C_1 and C_2 determined from Nikuradse's measurements, Equations (53) and (54) become:

$$\frac{v}{\sqrt{\frac{\tau_o}{\rho}}} = 5.5 + 5.75 \log_{10} \left(\sqrt{\frac{\tau_o}{\rho}} \frac{y}{\nu}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (55)$$

and,

$$\frac{v}{\sqrt{\frac{\tau_o}{\rho}}} = 5.85 + 5.75 \log_{10} \frac{y}{\epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots (56)$$

in which the value, 5.75, corresponds to $\kappa = 0.40$, and includes the conversion factor, 2.3026, between the natural and common systems of logarithms.

Writing each of these equations in terms of the maximum velocity at the distance, $y = r_o$, from the wall:

$$\frac{v_{max}}{\sqrt{\frac{\tau_o}{\rho}}} = 5.5 + 5.75 \log_{10} \left(\sqrt{\frac{\tau_o}{\rho}} \frac{r_o}{\nu}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (57)$$

and,

$$\frac{v_{max}}{\sqrt{\frac{\tau_0}{\rho}}} = 5.85 + 5.75 \log_{10} \frac{r_0}{\epsilon} \quad \text{.....(58)}$$

and subtracting Equations (55) and (56), respectively, there results the single general relationship for the velocity defect,

$$\frac{v_{max} - v}{\sqrt{\frac{\tau_0}{\rho}}} = 5.75 \log_{10} \frac{r_0}{y} \quad \text{.....(59)}$$

which is valid for the wall regions in either case, and, moreover, gives good results over the entire central portion of the flow (see Curve II in Fig. 15(*c*)).

The full significance of Equation (55) will be revealed by careful study of Figs. 16 and 17 based upon measurements by Nikuradse. In Fig. 16 the ratio

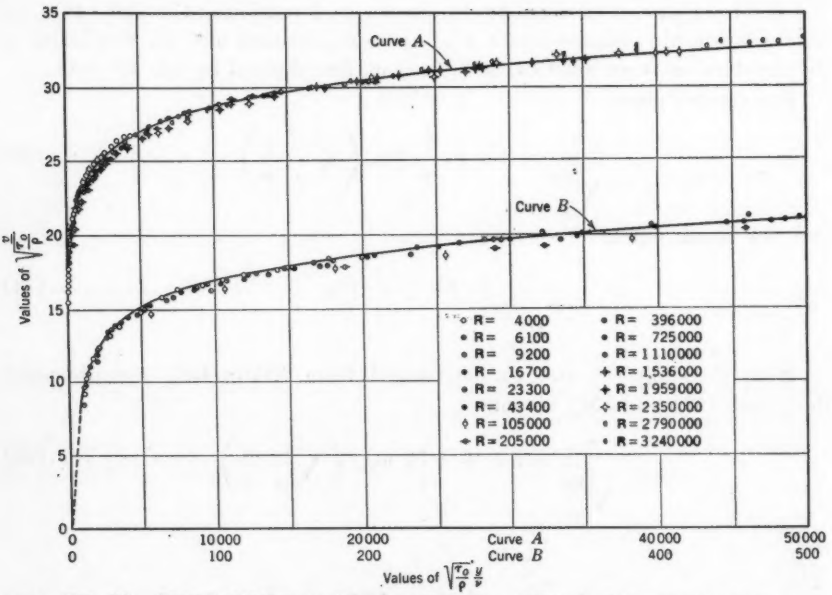

FIG. 16.—SEMI-LOGARITHMIC PLOT OF EQUATION (32) IN COMPARISON WITH MEASURED VALUES.

of velocity to friction velocity has been plotted as a function of the "friction-distance parameter", $\sqrt{\frac{\tau_0}{\rho}}\frac{y}{\nu}$, which is structurally equivalent to the Reynolds number; Fig. 17 is the same plot in semi-logarithmic form. In Fig. 16 the wall itself is represented by the ordinate axis, whereas the distance to the pipe center, $y = r_0$, is different for each different set of conditions, and depends entirely upon the magnitude of the friction-distance parameter. The horizontal scale of this plot has been given two different values, showing twice

the same general curve drawn through the plotted points; the more extended scale shows to good advantage the existence of the laminar film, in which the velocity distribution is practically linear.

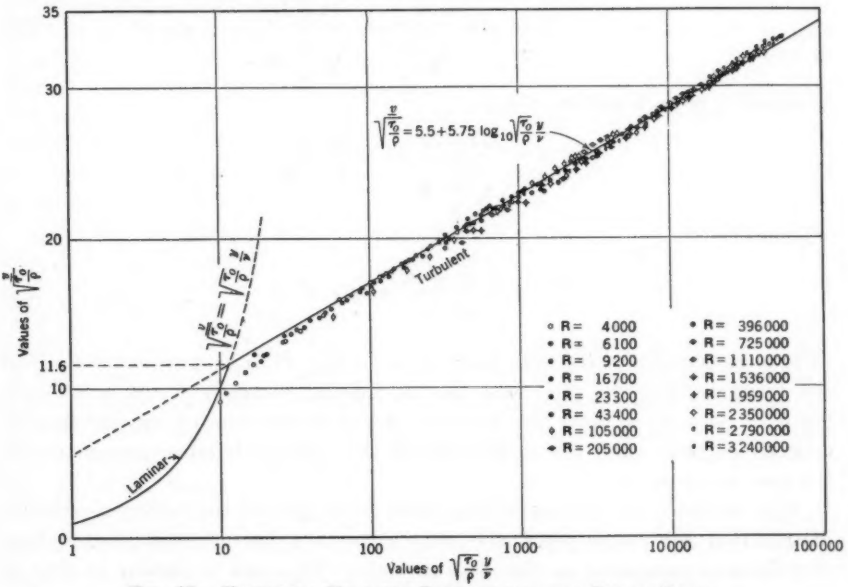

FIG. 17.—UNIVERSAL VELOCITY DISTRIBUTION FOR ROUGH PIPES.

Equation (55) plotted in semi-logarithmic form (Fig. 17) must obviously be a straight line. Points measured by Nikuradse on smooth pipes for various values of R are seen to fall upon this linear plot throughout an extreme range of the friction-distance parameter. Even at the pipe centers (the last groups of plotted points) only a slight deviation is noticeable. The departure in the wall region is obviously the result of viscous action as the boundary film is approached; this may be further clarified by constructing the curve of laminar distribution in this film, in which the following equality will be seen to hold:

$$\frac{v}{\sqrt{\frac{\tau_0}{\rho}}} = \sqrt{\frac{\tau_0}{\rho}}\frac{y}{\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (60)$$

since $v = \frac{\tau_0 y}{\mu}$. Thus, the lowest measured points are seen to be in the transition zone between the laminar and the turbulent regions. Not only may the approximate thickness of the laminar film be found from this plot, but also the extent of the viscous influence in the turbulent region. By combining Equations (55) and (60), at the point of intersection,

$$\frac{v}{\sqrt{\frac{\tau_0}{\rho}}} = \sqrt{\frac{\tau_0}{\rho}}\frac{y}{\nu} = 11.6 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (61)$$

Since this intersection of the laminar and turbulent velocity distribution curves may be chosen arbitrarily as the border of the laminar film, y will then equal δ, and the thickness of the laminar film will be given by,

$$\delta = 11.6 \frac{\nu}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{11.6\, D}{\sqrt{\frac{f}{8}}\, \mathbf{R}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (62)$$

Comparing with Equation (36),

$$\delta = \frac{11.6\, D}{\sqrt{\frac{f}{8}}\, \mathbf{R}} = 8 \frac{v_w}{V} \frac{D}{f\, \mathbf{R}}$$

whence,

$$\frac{v_w}{V} = \frac{11.6 \sqrt{f}}{\sqrt{8}} = 4.1 \sqrt{f} \quad \ldots\ldots\ldots\ldots\ldots\ldots (63)$$

It is to be noted that the only fixed point in Fig. 17 is this arbitrary limit of the laminar film; the pipe wall lies an infinite distance to the left (since $\log 0 = -\infty$), whereas the location of the center line of any given pipe depends entirely upon the magnitude of the friction-distance parameter for the case in question.

Fig. 18 shows the corresponding logarithmic plot of the universal velocity distribution for rough pipes, the ratio of y to ϵ having replaced the friction-distance parameter as the abscissa scale. This plot is similar to that of Fig. 17 in all respects, except for the absence of the laminar boundary film

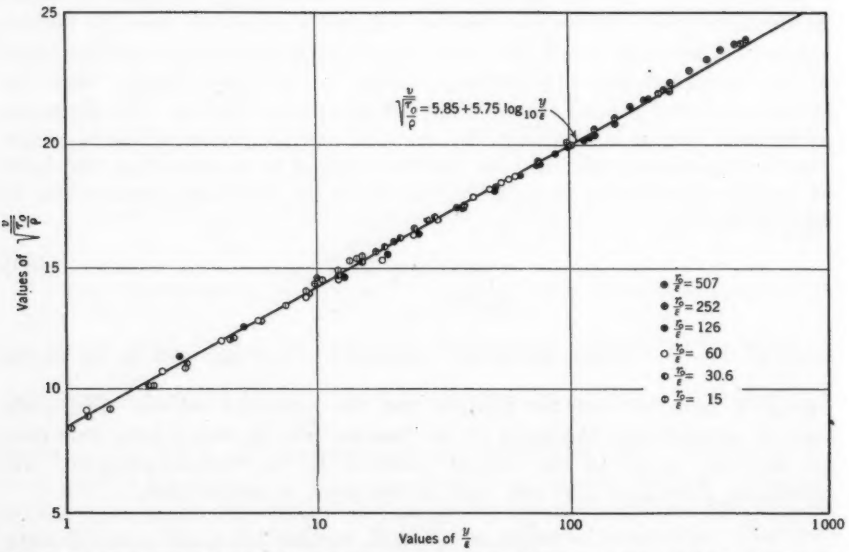

FIG. 18.—THE VON KÁRMÁN-NIKURADSE RESISTANCE FORMULA FOR SMOOTH PIPES IN COMPARISON WITH EXTRAPOLATED CURVES OF OTHER INVESTIGATORS.

and the accompanying viscous influence in the wall region. It will be noted that points determined by Nikuradse for various values of relative roughness fall upon a straight line even as close to the wall as the outer edges of the roughness particles (where $y = \varepsilon$). Moreover, inspection of Equations (55) and (56) will show a striking similarity between the respective constants. It will be seen that, since the factor, 5.75, is the same in both expressions, the two curves have identical slopes and are simply shifted on the plot by the difference between 5.5 in the one case and 5.85 in the other. Different forms of roughness, however, will surely require a change in one or both constants; at any rate, the term, ε, must then denote far more than merely the extent of linear projection from the wall.

The fact has already been mentioned that the value, 5.75, corresponds to $\kappa = 0.40$ (combined with the conversion factor between the two logarithmic systems). If Equations (52) and (59) are correct for the regions for which they were developed, then they must coincide in the zone where the two regions overlap. This they will never do, however, so long as the same value of κ is used for both—as must be done if this is a true universal constant. This explains more fully why von Kármán found that κ must be about 0.36 for the central regions, and Nikuradse was forced to choose the higher magnitude of 0.40 for the outer portions of the flow. The two curves plotted in Fig. 15(*c*) are based upon these two different values of κ.

Furthermore, owing to the linear characteristics of Equations (55) and (56) when plotted in semi-logarithmic form, it will be evident that the velocity distribution curves computed from these relationships must have a finite slope at the pipe axis, although actual velocity curves all reach a zero slope at this point. While the discrepancy is so slight as to be entirely negligible in practice, it points to a fallacy in the fundamental assumptions upon which these expressions were based. That is, the application of Equation (45) to the entire central region of flow presumes a constant correlation of the two essential components of velocity fluctuation; actually, there is no reason to expect such correlation near the pipe axis, for points in this vicinity are practically equi-distant from either side of the pipe wall. If the assumed variation of the mixing length as expressed in Equation (45) is plotted after the manner of Fig. 14, these computed values will follow the actual curve over the greater portion of the abscissa scale, but near the center line will drop away rapidly, approaching the value of zero at the axis. Obviously, the analytical study of the velocity distribution has not quite reached perfection.

Since similarity of the turbulent mechanism has been found to result in a general function for the velocity distribution, the close relation existing between velocity distribution and shear would indicate that there is also a universal function for the resistance coefficient. Equation (55) written for the maximum velocity at the distance, $y = r_o$, and in terms of the universal constant κ will have the form:

$$\frac{v_{max}}{\sqrt{\frac{\tau_o}{\rho}}} = 5.5 + \frac{1}{\kappa} \log_e \left(\sqrt{\frac{\tau_o}{\rho}} \frac{r_o}{\nu} \right) \quad \text{.....(64)}$$

As shown in Fig. 15(*c*), furthermore, the general nature of the velocity-defect curve must permit a direct computation of the difference between the maximum and mean velocities in terms of the common denominator, $\sqrt{\frac{\tau_0}{\rho}}$. This is nothing more than the average ordinate of the velocity-defect curve, when the latter is considered the cross-section of a surface of revolution about the center line of the pipe. The average ordinate of Curve II, therefore, may be determined as follows, as shown by Professor Bakhmeteff, through integration of Equation (59):

$$\frac{v_{max} - V}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{\int_{r_0}^{0} 2\pi (r_0 - y) \frac{1}{\kappa} \log_e \frac{r_0}{y} dy}{\pi r_0^2} = \frac{3}{2\kappa} = 3.75 \ldots (65)$$

Although Nikuradse found that a value of 4.07 would better fit experimental measurements of v_{max} and V, Equation (65) will be used without modification in the following development; it is this value which has been plotted as the mean in Fig. 15(*c*).

Subtracting Equation (65) from Equation (64), there results an expression for the average velocity of flow as a function of the friction velocity, the universal constant, and the friction-distance parameter:

$$\frac{V}{\sqrt{\frac{\tau_0}{\rho}}} = -3.75 + 5.5 + \frac{1}{\kappa} \log_e \left(\sqrt{\frac{\tau_0}{\rho}} \frac{r_0}{\nu} \right) \ldots\ldots\ldots (66)$$

Equation (66) may now be reduced to a more significant form through introduction of Equation (13) in the forms,

$$\frac{V}{\sqrt{\frac{\tau_0}{\rho}}} = \sqrt{\frac{8}{f}} \text{ and } \sqrt{\frac{\tau_0}{\rho}} = V \sqrt{\frac{f}{8}}:$$

$$\frac{\sqrt{8}}{\sqrt{f}} = 1.75 + \frac{1}{\kappa} \log_e \left(\frac{\sqrt{f}}{\sqrt{8}} \frac{VD}{2\nu} \right)$$

$$= 1.75 + \frac{l}{\kappa} \log_e \frac{1}{4\sqrt{2}} + \frac{1}{\kappa} \log_e \left(R \sqrt{f} \right)$$

Hence,

$$\frac{1}{\sqrt{f}} = -0.91 + 2.03 \log_{10} \left(R \sqrt{f} \right) \ldots\ldots\ldots (67)$$

Thus, a very simple relationship for the resistance factor in terms of the universal constant and the Reynolds number has been developed directly from the general expression for the velocity distribution. This basic relationship was first derived by von Kármán in terms of the maximum velocity, and

later put into the form of Equation (67) by Prandtl. With slight modification of constants according to the experimental data of Nikuradse, this basic equation finally becomes:

$$\frac{1}{\sqrt{f}} = -0.8 + 2 \log_{10} (R \sqrt{f}) \quad \text{.....(68)}$$

Inasmuch as Equation (68) is based upon physical analysis, it might be expected to be valid (with correct constants) from $R = 100\,000$ to extremely high, if not to infinite, values of R. The only drawback to its use for general purposes is the appearance of the square root of f on both sides of the equation. Although this disadvantage may be offset through use of a plotted curve, Nikuradse has suggested the empirical Equation (35), which is quite accurate for all ordinary values of R above the Blasius range, and which, moreover, may be extrapolated safely a considerable distance beyond the present limit of experimental measurements. As seen from Fig. 19, in which

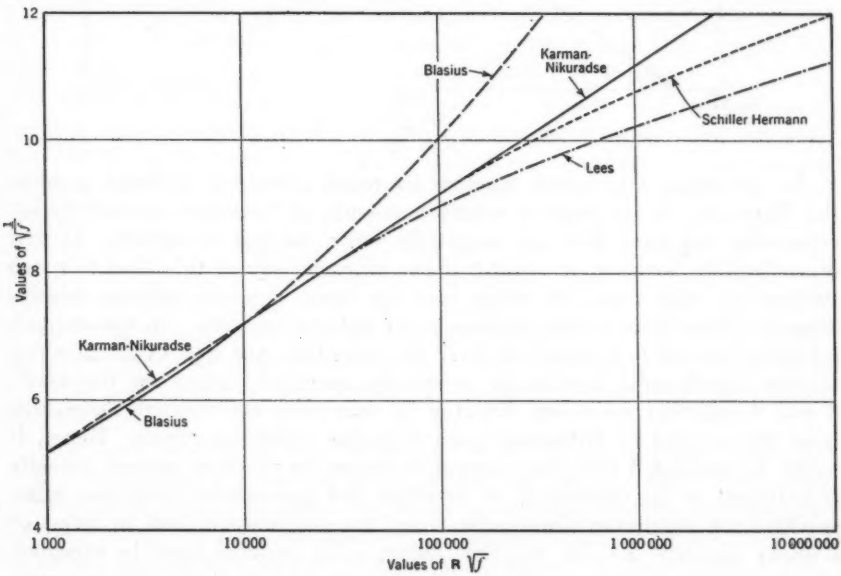

FIG. 19.

the curves of Equations (68) and (35) are practically identical, previous empirical curves deviate considerably from the linear form.

A similar operation performed on Equation (56), with slight correction of constants to conform with Nikuradse's data, results in the expression already given for the resistance in rough pipes (see Equation (38)):

$$\frac{1}{\sqrt{f}} = 1.74 + 2 \log_{10} \frac{r_0}{\epsilon}$$

The excellent agreement between Nikuradse's measurements and the linear quality of the analytical expression is shown in Fig. 20. Once again the

curves for smooth and rough pipes are seen to have the same slope, differing only in position as determined by the constants, — 0.8 and 1.74, at least for the type of roughness studied by Nikuradse.

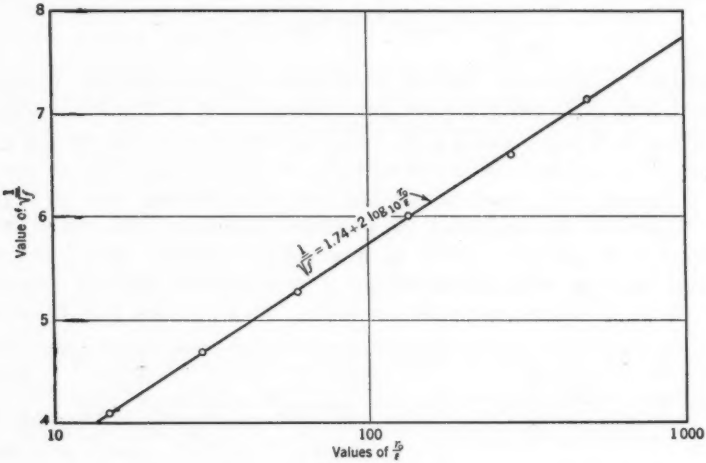

FIG. 20.

In developing a universal function for rough pipes, one is forced to adopt the dimension, ϵ, for want of a better measure of roughness characteristics. Obviously, this term does not completely define the type of surface. It may prove feasible, however, to adapt a linear characteristic of this kind to rough surfaces of other types, in which case the term, ϵ, would indicate relative behavior rather than an actual measure of radial projection. In this respect, reference is made to a paper[21] by V. L. Streeter, Jun. Am. Soc. C. E., in which further experimental results for artificially roughened pipes are described; it was found that roughness differing in both form and relative magnitude from that studied by Nikuradse yielded similar resistance curves. Hence, it might be concluded that this general function for the flow pattern actually is universal in its application to artificial and commercial roughness alike, provided the roughness characteristic may be adequately defined in terms of a linear quantity, ϵ. For the time being, wavy surfaces must be excepted, of course, since the form of the function may not be the same. At any rate, it has been proved conclusively that roughness dimensions alone are not sufficient indication of the resistance to flow, but must always be considered with relation to the dimensions of the pipe.

Mention must be made of the fact that, whereas the mathematical genius of von Kármán was first responsible for the development of these universal functions, several other scientists of note have deduced similar relationships in various ways; Prandtl[22] has continued to hold a foremost place in this field and it was under the latter's directorship of the Kaiser Wilhelm Institut für

[21] "Frictional Resistance in Artificially Roughened Pipes", *Proceedings* Am. Soc. C. E., February, 1935, p. 163.

[22] "Neuere Ergebnisse der Turbulenzforschung", *Zeitschrift* des Vereines Deutscher Ingenieure, Band 77, Nr. 5, 1933.

Strömungsforschung, in Göttingen, that Nikuradse conducted his extensive and systematic measurements of turbulent flow in pipes. Also from the Prandtl school has come the latest effort to study the problem of turbulence in the light of probability and statistical physics.[23]

Although such experimental data as herein cited were limited by laboratory conditions so far as wall surface and pipe diameter are concerned, a splendid beginning has been made in the experimental verification of such broad analytical deductions.

Research of the Future

Throughout this paper constant emphasis has been laid upon the necessity of systematic, physically sound investigation of fluid motion. Various schools in Europe and several of note in the United States are engaged upon such fluid research of an advanced nature, but both co-operation and correlation of effort are essential to continued progress. It is noteworthy that within the past year (1934–35), at the instigation of the World Power Conference, a committee on fluid resistance has been created in the United States with the object of planning and sponsoring organized research in this field.

What problems still remain unsolved in the domain of fluid resistance should be quite apparent. Fluid motion may no longer be classified satisfactorily simply as laminar or turbulent flow, since turbulence is not always free from appreciable viscous resistance to flow. Whereas on the one hand laminar or purely viscous motion is practically a closed book so far as general pipe resistance is concerned, on the other hand, the problem of purely turbulent motion at very high Reynolds numbers is not yet fully solved; a complete rational analysis of the inner mechanism of turbulent movement is still to be achieved, despite the success of the methods herein described; and no successful attempt has been made as yet to analyze the combined effect of viscous and turbulent shear within the lower range in which the Blasius equation applies.

Of primary interest to the engineer, of course, is the correlation between artificial roughness as produced in the laboratory, and the actual roughness of commercial pipe, if only for the range of constant resistance coefficient. Not only does this leave unexplained the entire transition region between the smooth and rough régimes, but it fails completely to consider that type of pipe of which the behavior does not fall in either category—the pipe with "wavy" surface. It is obvious that neither the existing data on commercial pipe nor the recent experiments on a few types of artificial roughness provides sufficient evidence for the formulation of general relationships including every type of roughness under every possible condition of flow.

Once the realm of pipe resistance is left, the problem becomes even more complex. Open channels present a further variable in the form of the free surface, and the question of geometrical similarity of the profile and cross-section becomes of major importance; it is still a debatable question whether the hydraulic radius may continue to prove satisfactory as a general length

[23] "Turbulenz", von H. Gebelein, Springer, Berlin, 1935.

parameter. Doubtless, however, open channel flow is of so highly turbulent a nature that viscous influences are probably quite negligible; yet this again introduces a question in the mind of the experimenter, for open flow is often simulated in the laboratory by small-scale models, in which the viscous action is most assuredly of appreciable magnitude—even far above the laminar régime.

In other fields of engineering similar problems must be encountered, for the designers of air and water craft are in much the same difficult situation in their efforts to understand the complexities of skin and form resistance. This very universality of resistance phenomena may well continue to prove an impetus toward the advancement of knowledge in this field.

Acknowledgment

For many hours of stimulating discussion and criticism during the preparation of this paper, the writer expresses his sincere gratitude to Boris A. Bakhmeteff, M. Am. Soc. C. E.

APPENDIX

List of General Symbols

M, L, T = mass, length, and time as general dimensional terms.
F = any acting force.
I = generally, the inertial reaction to any acting force; specifically, the inertial character of flow; that is, the momentum passing a section per unit of time.
m = mass.
t = time.
V = the average velocity of flow for the entire cross-section.
v = in general, any instantaneous velocity vector; for pipe study, the temporal average velocity at any point in the cross-section.
$v_{max.}$ = the maximum value of v, occurring at the pipe axis.
p = pressure intensity.
R = Reynolds number.
f = resistance coefficient for pipes.
n = exponent in the general equation, $f = C\ \mathsf{R}^{(n-2)}$.
h = manometric head.
L = length of pipe.
D = diameter of pipe.
r_0 = radius of pipe.
r = radial distance from pipe axis.
y = normal distance from pipe wall.
ρ = density of fluid, or mass per unit of volume.

γ = specific weight of fluid, or weight per unit of volume.

g = gravitational acceleration; $g = \frac{\gamma}{\rho}$.

μ = viscosity of a fluid.

ν = kinematic viscosity of a fluid; $\nu = \frac{\mu}{\rho}$.

τ = intensity of shear at any point in the cross-section.

τ_0 = intensity of shear at the pipe wall.

ε = radial dimension of roughness particles.

δ = thickness of the boundary layer.

l = the mixing length.

κ = a universal constant relating to flow at high values of R.

ϕ = a function of.

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PAPERS

COMPARISON OF SLUICE-GATE DISCHARGE IN MODEL AND PROTOTYPE

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SYNOPSIS

There is a need for experimental data to support hydraulic model theory because confidence in the use of models hinges upon this checking of model experimental results with prototype experiments. It is seldom possible to check model experiments with prototype experiments although this condition has shown considerable improvement in the last few years. In this paper the aim is to present the results of experiments on the prototype and the model of sluice-gates.

NOTATION

The symbols used in this paper are defined, as follows:

c = a coefficient; c_d = a coefficient of discharge;
g = acceleration due to gravity;
l = scale ratio of length = $\frac{L_n}{L_m}$;
m = a subscript denoting model;
n = an exponent in Equation (1); as a subscript, n denotes "in nature" or "prototype";
q = scale ratio of flow ($q = \frac{Q_n}{Q_m}$);
v = velocity;
A = area;
F = Froude's number;
H = head;
L = length;
Q = rate of flow; Q_n = rate of flow for prototype; Q_m = rate of flow for model.

INTRODUCTION

A thorough search of technical literature reveals that the United States Bureau of Reclamation has made several experiments on canal head-gates and that the Ministry of Public Works of Egypt has made very thorough pro-

NOTE.—Discussion on this paper will be closed in April, 1936, *Proceedings*.

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totype and model experiments on the Assuan Dam on the Nile River. The experiments[2] made by the U. S. Bureau of Reclamation were on the prototype only. The experiments on the Assuan Dam[3] were continued for a period of 20 yr and include experiments on sluices 2 m wide by 3.5 m high (6 ft $6\frac{3}{4}$ in. by 11 ft $5\frac{3}{4}$ in.) and 2 m wide by 7 m high (6 ft $6\frac{3}{4}$ in. by 22 ft $11\frac{1}{2}$ in.) through the dam under heads (measured from the gate-sill) from 0.3 to 13.5 m (1.0 ft to 44.3 ft) for the smaller sluices, ten of which discharged into a large tank where the water was measured volumetrically. Six models were built of these smaller sluices and approximately 1 500 experiments were made. The comparison of model and prototype results for all experiments shows that the ratio of $\frac{Q_n}{Q_m}$ varies as $l^{2.5}$, or that Froude's model law will apply to discharge through sluices. The mean departure of $\frac{Q_n}{Q_m}$ from $l^{2.5}$ was found to range from $-$ 4.6% to $+$ 4.5%, with an average departure of $+$ 0.4 per cent.

Prototype

Plans of the Tremont gates at Lowell, Mass., with copies of data on experiments performed on these gates in June, 1868, by the late James B. Francis, Past-President and Hon. M. Am. Soc. C. E., were made available by Arthur T. Safford, M. Am. Soc. C. E. These gates were set in a canal and were apparently calibrated by Mr. Francis in order that they might be used to measure the quantity of water passing them. There are two sluices, of ashlar masonry, 9 ft wide by 7.12 ft high, to the top of the arch (see Fig. 1). Each sluice is closed by wooden gates 10 in. thick; the floors are set above the bottom of the canal; and the openings are flared. When in operation both gates were opened the same amount and the head-water was maintained at about the level of the top of the sluice. The head-water elevation was measured in a well at the Tremont cotton house after being conveyed from the apron by a $1\frac{1}{2}$-in. pipe through one of the sluices (see Fig. 1). The tail-water elevation was measured on a wooden gauge in the same well. The actual quantity of water passing the gates was measured with a rectangular weir and computed by means of the Francis weir formula. Correction for velocity of approach was made to the head measured at the weir, but such correction was not made to the head measured at the gates. This latter omission on the part of Mr. Francis was undoubtedly due to the fact that the gates were calibrated for practical purposes only. As this was the manner in which head would be measured for future discharge measurements, no error would result in computing discharges.

The results obtained by Mr. Francis were plotted on logarithmic paper. It will be noted in Fig. 2 that the resulting straight lines are not quite

[2] "Discharge Coefficients for Canal Head-Gates", by J. S. Longwell and Julian Hinds, Members, Am. Soc. C. E., *Reclamation Record*, Vol. 10 (1919), p. 475; and "Rating Curves for Canal Headgates", by Julian Hinds, M. Am. Soc. C. E., *Reclamation Record*, Vol. 13 (1922), p. 98.

[3] *Minutes of Proceedings*, Inst. C. E., Vol. 212, Pt. II (1920–21), p. 228; also, Vol. 218, Pt. II (1923–24), pp. 113 and 72.

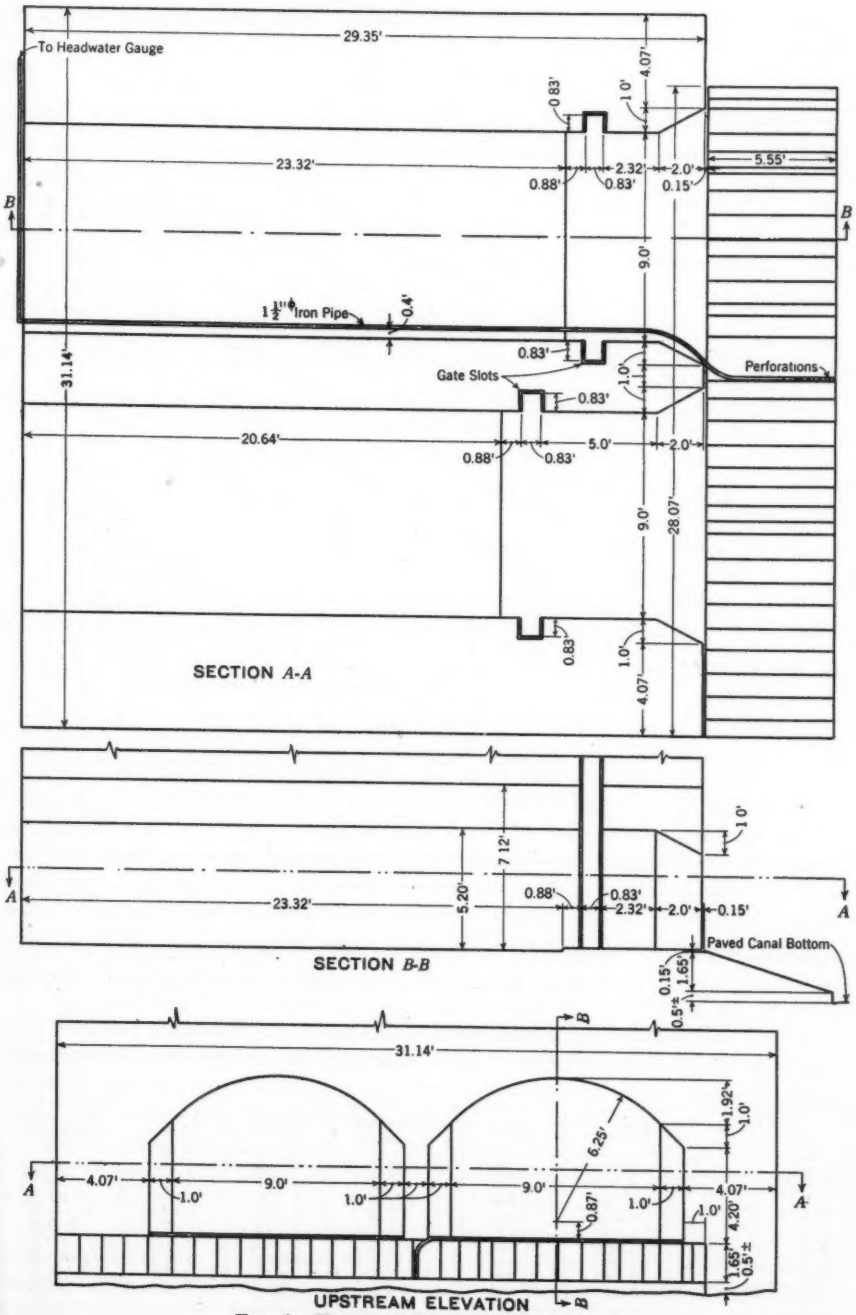

FIG. 1.—TREMONT GATES, LOWELL, MASS.

parallel for the several gate-openings, which would indicate that n in the formula,

$$Q = c_d A \sqrt{2g} H^n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

is not exactly equal in each case. The coefficient, c_d, is also not equal for all

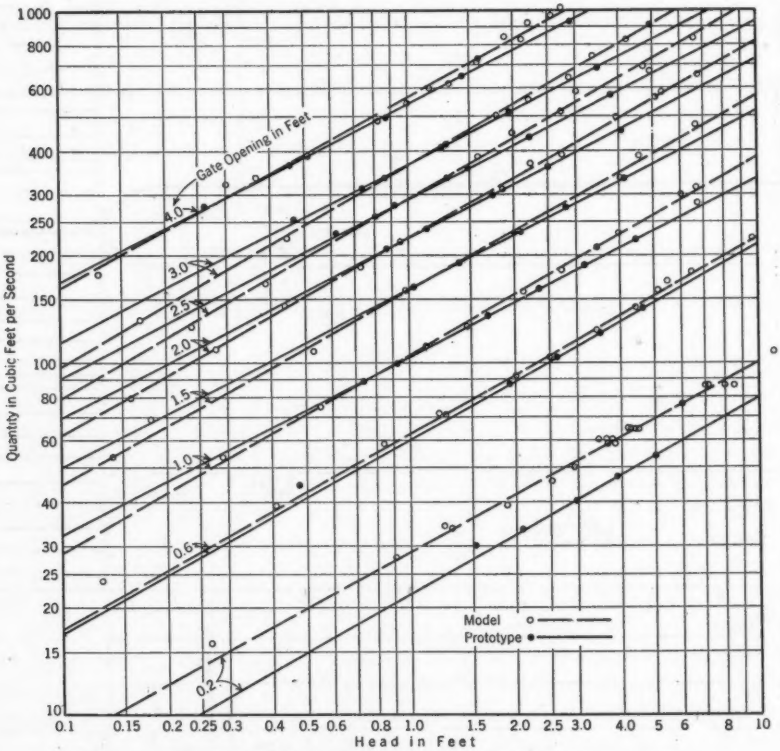

FIG. 2.—DISCHARGE CURVES; MODELS AND PROTOTYPE EXPERIMENTS.

openings or heads. Reference is made to Table 1 for a comparison of values of n and c_d. Table 2 gives values of c_d for various heads and gate-openings.

TABLE 1.—CURVE CONSTANTS, c_d AND n, IN EQUATION (1)

Gate-opening, in feet	n			c_d		
	Model	Nature	Percentage difference	Model	Nature	Percentage difference
0.2	0.537	0.557	−3.6	1.007	0.762	+32.2
0.6	0.558	0.551	+1.3	0.721	0.698	+3.3
1.0	0.562	0.505	+11.3	0.723	0.716	+1.0
1.5	0.550	0.510	+7.8	0.736	0.740	−0.5
2.0	0.560	0.508	+10.2	0.778	0.771	+0.8
2.5	0.565	0.506	+11.7	0.804	0.800	+0.5
3.0	0.570	0.504	+13.1	0.838	0.841	−0.4
4.0	0.543	0.506	+7.3	0.971	0.933	+4.1
Mean *	0.558	0.506	+9.0	0.808	0.800	+1.3

* 0.2-ft gate-opening omitted.

TABLE 2.—COEFFICIENT OF DISCHARGE, c_d, IN EQUATION (1) ($n = 0.5$)

Head, in feet	Description	Gate-Opening, in Feet								
		0.2	0.6	1.0	1.5	2.0	2.5	3.0	4.0	Mean*
0.5	Model	0.980	0.699	0.695	0.713	0.745	0.765	0.795	0.950	0.777
	Nature	0.735	0.672	0.715	0.739	0.770	0.800	0.834	0.932	0.798
	Percentage difference	+33.3	+4.0	−2.8	−3.5	−3.2	−4.4	−4.7	+1.9	−1.8
0.0	Model	0.997	0.721	0.727	0.739	0.775	0.803	0.832	0.968	0.807
	Nature	0.762	0.693	0.720	0.739	0.775	0.803	0.836	0.934	0.801
	Percentage difference	+30.8	+4.0	+1.0	0.0	0.0	0.0	−0.3	+3.6	+1.3
1.5	Model	1.017	0.735	0.746	0.755	0.797	0.833	0.857	0.990	0.830
	Nature	0.777	0.717	0.718	0.744	0.777	0.808	0.838	0.934	0.803
	Percentage difference	+30.8	+2.5	+3.9	+1.5	+2.6	+3.1	+2.3	+6.0	+3.1
2.0	Model	1.028	0.755	0.760	0.768	0.808	0.843	0.873	1.004	0.843
	Nature	0.797	0.722	0.725	0.745	0.785	0.805	0.850	0.931	0.807
	Percentage difference	+29.0	+4.6	+4.8	+3.1	+2.9	+4.7	+2.7	+7.8	+4.5
2.5	Model	1.042	0.767	0.767	0.775	0.818	0.860	0.890	1.010	0.853
	Nature	0.801	0.730	0.724	0.746	0.785	0.807	0.846	0.943	0.808
	Percentage difference	+30.1	+5.1	+6.0	+3.9	+4.2	+6.6	+5.2	+7.1	+5.4
3.0	Model	1.050	0.766	0.775	0.784	0.830	0.865	0.905	1.020	0.863
	Nature	0.820	0.734	0.724	0.752	0.780	0.808	0.847	0.940	0.808
	Percentage difference	+28.0	+4.4	+7.0	+4.3	+6.4	+7.1	+6.9	+8.5	+6.3
3.5	Model	1.055	0.778	0.790	0.790	0.833	0.875	0.915	.....	0.841
	Nature	0.825	0.741	0.722	0.745	0.778	0.807	0.840	.....	0.788
	Percentage difference	+27.9	+5.0	+9.4	+6.0	+7.1	+8.4	+8.9	.....	+7.5
4.0	Model	1.063	0.786	0.788	0.790	0.848	0.878	0.923	.....	0.845
	Nature	0.830	0.752	0.727	0.750	0.783	0.810	0.842	.....	0.782
	Percentage difference	+28.1	+4.5	+8.4	+5.3	+8.3	+8.4	+9.6	.....	+7.4
4.5	Model	1.070	0.792	0.797	0.799	0.850	0.890	0.925	.....	0.852
	Nature	0.834	0.760	0.720	0.745	0.785	0.810	0.850	.....	0.782
	Percentage difference	+28.3	+4.2	+10.7	+7.2	+8.3	+9.9	+8.8	.....	+8.2
5.0	Model	1.068	0.794	0.799	0.805	0.860	0.893	0.933	.....	0.858
	Nature	0.836	0.763	0.727	0.746	0.782	0.812	0.845	.....	0.782
	Percentage difference	+27.8	+4.1	+9.9	+7.9	+10.0	+10.0	+10.4	.....	+8.7
Mean	Percentage difference	+30.4	+4.2	+5.8	+3.6	+4.7	+5.4	+5.1	+5.8	+5.1

*0.2-ft gate-opening omitted.

CHANNEL

The channel in which the model of the gates was set, was constructed especially for this experiment. It consisted of four sections—a weir tank in which the water was measured; an entrance box in which the turbulence of the water due to passing over the weir was absorbed; the channel proper with plate-glass sides; and the exit box. The weir tank was 3 ft 6 in. square, 5 ft deep, and contained a sharp-edged weir 1 ft deep, with a 90° notch, which was calibrated by means of volumetric tanks immediately before being put into use on this experiment.

The water flowed horizontally, and with marked turbulence, from the weir tank into the entrance box. A vertical baffle placed near the exit of the weir box deflected the water downward and dissipated all the horizontal velocity. Another baffle set in the entrance box served to accelerate the water gradually so that it entered the channel with a minimum of turbulence.

The channel was 2 ft wide, and the bottom (a 4-in. non-reinforced concrete slab, 5 ft 3 in. long, supported on two 3-in. 5.7-lb I-beams) was set 3 ft 5 in. above the floor of the laboratory. It was designed so that there would be practically no deflection under the heaviest possible loading. The sides consisted of $\frac{3}{4}$-in. plate-glass windows supported in structural steel frames. A head measuring gauge, reading to 0.01 cm, ran on adjustable rails, $1\frac{1}{16}$ in. in diameter, which were supported by $\frac{3}{8}$-in. bolts and double nuts

spaced 9 in., center to center, from the steel framework for the glass channel sides. This gauge was movable in three directions so that measurements could be obtained in any part of the channel. No deflection was observed in the carriage rails when the gauge was in operation.

Water is collected in the outlet box, from which it is conveyed back to the main storage reservoir. It passes from the reservoir through a centrifugal pump and into a skimming tank in which a constant surface elevation is maintained under all conditions. Finally, it is distributed from this point to all parts of the laboratory under conditions insuring a constant rate of flow at all times.

Model

A scale ratio of 1 to 15 was determined upon as being the most satisfactory. The model was constructed in two parts which were grouted together after placing the model in the channel. A base section comprising that part of the model up to the floors of the sluices was constructed of sand-cement mortar. The top section containing the sluices and gates was constructed of cement and white pine sawdust. This use of sawdust resulted in a model approximately one-third as heavy as the sand-cement mixture, and no noticeable swelling or shrinking occurred after it had set. The model was smoothed (not polished) with an emery stone. The gates were of redwood, 0.66 in. thick; they were square-edged, and were held in place by wedges.

The elevation of the up-stream water surface was measured, as in the prototype (see Fig. 1), by means of a perforated pipe lying on the apron in front of the pier and leading to a glass well at the down-stream end of the sluices. The down-stream water-surface elevation was measured in a stilling-well, a vertical shutter arrangement controlling the tail-water elevation.

Experimental Procedure

In making an experimental test the gates were first set to the desired opening with the aid of the traveling gauge and were wedged in position. A predetermined rate of flow was allowed through the model, and the tail-water elevation was controlled by means of the shutters so that the head-water elevation would be at approximately the top of the sluices.

After conditions had become constant, readings were taken of water-surface elevations above and below the gates and at the measuring weir. Any unusual conditions of flow were noted. The water approached the model with little velocity and an absence of turbulence. After passing the gates there was considerable turbulence noticeable in the sluices accompanied by a horizontal surface roller near the gates. Under some conditions a hydraulic jump would form in the sluices and when this occurred the tail-water elevation was measured down stream from the jump.

Coefficient of Discharge

The experimental results are shown by broken lines in Fig. 2 and correspond to the solid curves for the prototype. For different gate-openings these curves are not parallel, as was the case with the curves for the prototype. This

is also demonstrated by Table 1 in that n is not a constant for all gate-openings. The value of n in Equation (1), as obtained from the model experiments, is seen to vary from + 1.3% to + 13.1% of the values obtained from the prototype experiments with an average deviation of + 9.0 per cent. The values of c_d obtained from the model experiments for use in Equation (1) can be seen to vary from − 0.5% to + 4.1% of the values obtained from the prototype experiments with an average deviation of + 1.3%, with the 0.2-ft gate-opening omitted. With $n = 0.5$ (which is the value commonly assigned to n) the values of c_d computed from the model experiments were found to vary from − 4.7% to + 10.7% of the values obtained from the prototype experiments with an average deviation of + 5.1 per cent.

It will be noted that the results obtained for the 0.2-ft gate-opening were omitted from the averages. This was done because these experiments support the recommendation of Messrs. Hurst and Watt[3] that,

"Until further experiments on models of other structures define the limiting conditions more closely, it will be well to keep the product of velocity, in centimeters per second, and the smallest dimension of the orifice, in centimeters, above, say, 100 [cm² per sec = 0.11 ft per sec], and in general not to use orifices of less than 3 centimeters [1⅛ in.] in their smallest dimension."

The 0.2-ft gate-opening corresponds to a 0.41-cm ($\frac{5}{32}$-in.) model gate-opening and the velocity through the gates for a 5-ft head is 151 cm per sec (5 ft per sec) which gives 62 as the product of velocity and least dimension for the highest head. For the 0.6-ft gate-opening, the model gate-opening is 1.22 cm ($1\frac{5}{32}$ in.) and the velocity is 31 cm per sec, or 1 ft per sec ($H = 0.5$ ft) which gives 38 as the product of velocity and smallest opening. In spite of the fact that the 0.6-ft gate-opening had values of gate-opening times velocity of considerably less than 100, the results were included in the averages because of the good agreement between model and prototype results.

The model gate-openings are not greater than 3 cm (1⅛ in.) until the 1.5-ft prototype gate-opening is reached, and the product of velocity, in centimeters per second, and the smallest dimension, in centimeters, for all heads is not greater than 100 until this same gate-opening is reached. Because of the fairly good agreement of the results obtained for model and prototype, it was felt that all results for gate-opening of 0.6 ft, or larger, should be included in the averages. The average percentage deviation of values of c_d (see Fig. 3(a)), as obtained from model experiments, from values of c_d obtained from prototype experiments for different heads, was found to increase as the head increases. The reason for this increase is perhaps due to some peculiarity of the sluice which was not copied exactly or to the lessening effect of friction as the velocity increases.

The recommendations of Messrs. Hurst and Watt[3] were checked roughly. Good results were obtained for smaller values of least dimension and the product of velocity and least dimension. The lowest satisfactory values of these items obtained in these experiments for all heads tested were 1.22 cm ($1\frac{5}{32}$ in.) and 28 cm² per sec (0.03 ft² per sec), respectively. Upon the basis of these experiments it would seem that the recommendations of Messrs. Hurst and Watt are conservative.

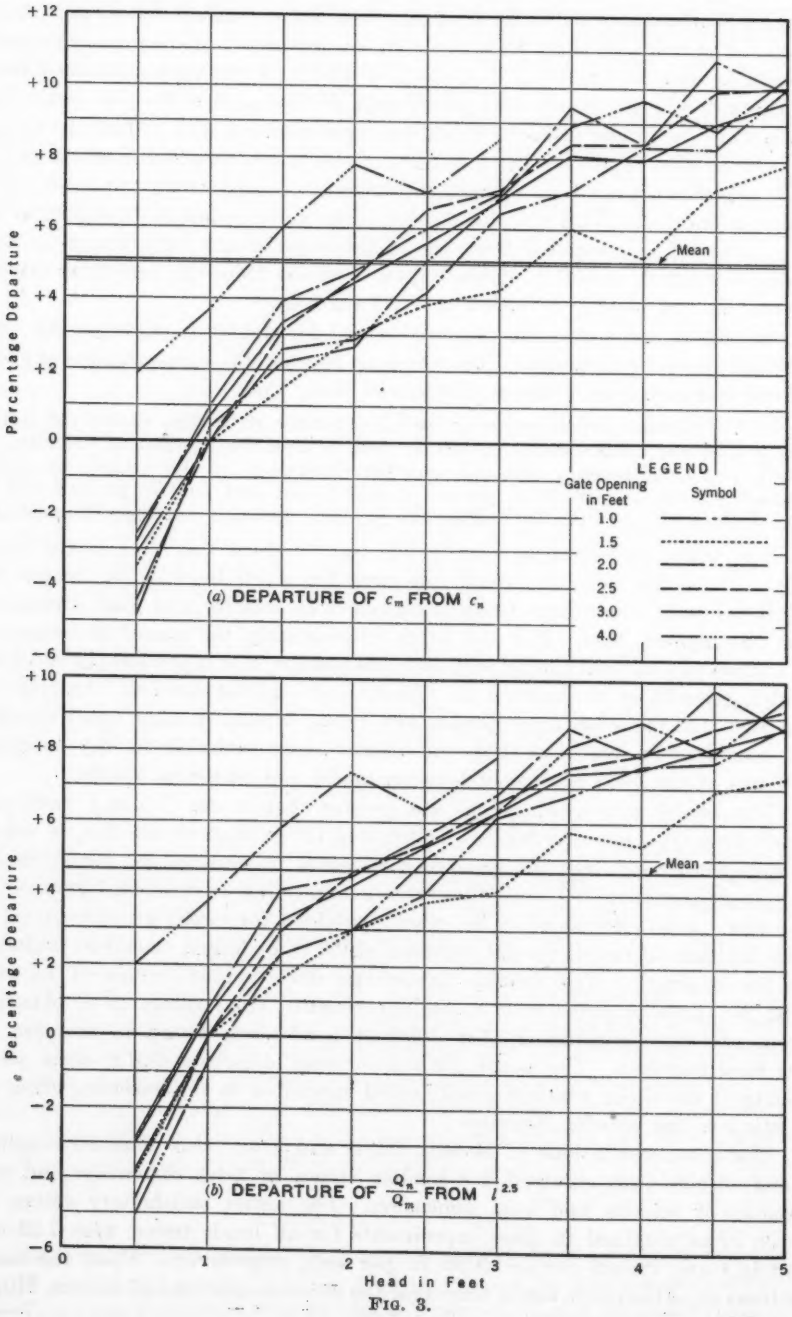
+12
+10
+8
+6
+4
+2
0
−2
−4
−6
Percentage Departure
Mean
LEGEND
Gate Opening in Feet
Symbol
1.0
1.5
2.0
2.5
3.0
4.0
(a) DEPARTURE OF c_m FROM c_n
(b) DEPARTURE OF $\frac{Q_n}{Q_m}$ FROM $l^{2.5}$
0
1
2
3
4
5
Head in Feet

FIG. 3.

Agreement with Froude's Law

For the engineer trying to interpret the results of experiments upon models of sluice-gates, the laws that apply, are important, as well as the probable deviation of the predicted discharge from the actual discharge of the prototype. The model law for the case of sluice-gates[4] is commonly attributed to Froude and applies when terrestrial gravitation is the predominating physical force and all other physical forces affecting flow can be neglected. These other physical forces do affect the flow, but the effect is so small that it is not considered. The dimensionless statement of Froude's law is:

$$\mathsf{F} = \frac{v^2}{L\,g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The factor for the conversion of model discharge to prototype discharge is:

$$\frac{Q_n}{Q_m} = \sqrt{\frac{g_n}{g_m}}\; l^{2.5} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Since $g_n = g_m$ for the case under consideration, the conversion factor is:

$$\frac{Q_n}{Q_m} = l^{2.5} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

For this model in which $l = 15$ the value of $l^{2.5}$ is 871 and, therefore, all model discharges were multiplied by 871 to obtain the probable prototype discharge.

Table 3 is a list of the percentage deviation of $\frac{Q_n}{Q_m}$ from $l^{2.5}$ as obtained from the curves of Fig. 2. The values of $\frac{Q_n}{Q_m}$ were found to differ from

TABLE 3.—Percentage Departure of $l^{2.5}$ from $\frac{Q_n}{Q_m}$

Head, in feet	Gate-Opening, in Feet								
	0.2	0.6	1.0	1.5	2.0	2.5	3.0	4.0	Mean*
0.5	+25.1	+3.7	−2.9	−3.9	−3.7	−4.6	−5.1	+2.0	−2.1
1.0	+23.7	+4.1	+0.7	+0.1	−0.1	0.0	−0.7	+3.7	+1.1
1.5	+23.7	+2.4	+4.1	+1.7	+2.4	+2.9	+2.3	+5.8	+3.1
2.0	+22.5	+4.1	+4.5	+3.0	+3.0	+4.6	+3.0	+7.4	+4.2
2.5	+23.1	+4.4	+5.6	+3.8	+4.0	+6.3	+4.9	+6.4	+5.1
3.0	+22.0	+4.4	+6.8	+4.1	+6.2	+6.6	+6.3	+7.8	+6.0
3.5	+22.0	+4.9	+8.7	+5.8	+6.8	+7.6	+8.2	.....	+7.0
4.0	+22.0	+4.4	+7.9	+5.3	+7.6	+7.9	+8.8	.....	+7.0
4.5	+22.0	+4.5	+9.8	+6.9	+7.7	+8.7	+8.0	.....	+7.6
5.0	+22.7	+4.0	+8.7	+7.3	+8.7	+9.1	+9.5	.....	+7.9
Mean	+22.9	+4.1	+5.4	+3.4	+4.2	+4.9	+4.5	+5.5	+4.7

* 0.2-ft gate-opening omitted.

$l^{2.5}$ from − 5.1% to + 9.8% with an average deviation of + 4.7 per cent. This is clearly shown in Fig. 3(*b*) where the deviation increases as the head increases.

[4] "Similitude in Hydraulic Models", by K. C. Reynolds, Assoc. M. Am. Soc. C. E., *Engineering News-Record*, Vol. 113, No. 8 (August 23, 1934), p. 238; and "Notes on Dimensional Analysis and the Principle of Similitude", by K. C. Reynolds (for private distribution). Also, "Hydraulic Laboratory Practice", Edited by the late John R. Freeman, Past-President and Hon. M. Am. Soc. C. E., pp. 759 and 775, A. S. M. E., N. Y., 1929.

Fig. 4 is a plot of Froude's number, F, as abscissa, against the coefficient of discharge, c, as ordinate. Froude's number was calculated from the equation,

$$\mathsf{F} = \frac{v^2}{g\,H} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

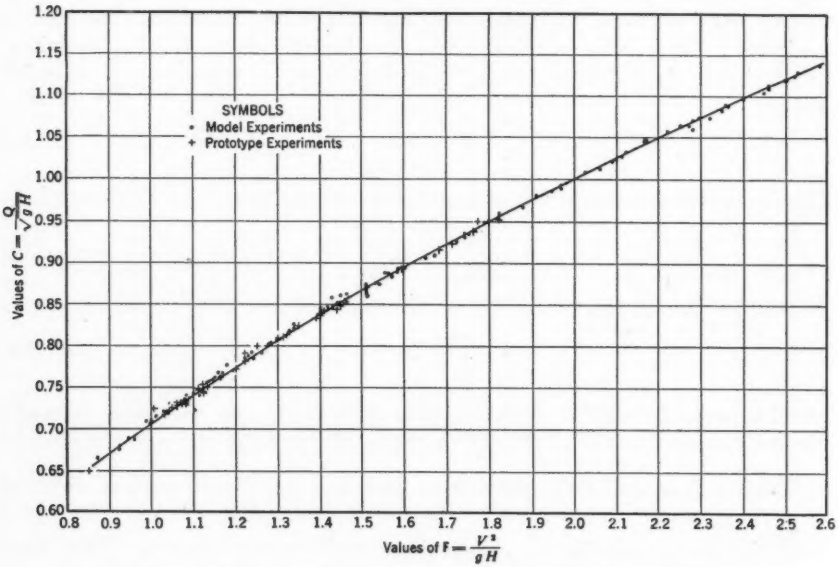

FIG. 4.—FROUDE'S NUMBER VERSUS COEFFICIENT OF DISCHARGE.

in which $v = \frac{Q}{A}$ is the actual velocity at the gate section and H is the total loss of head through the gates and sluices. The coefficient of discharge was calculated from the equation,

$$c_d = \frac{Q}{A\sqrt{2\,g\,H}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

in which Q is the actual rate of flow passing the gates and A is the area of gate-opening. Both these numbers are dimensionless and, therefore, have the advantages inherent in pure numbers.

This plot is similar to the plot of the friction factor, f, and Reynolds' number, R, for pipes[5]. The curves plotted for several pipes of different diameters and the same roughness, should follow, closely, an average curve since all circular pipes can be said to be geometrically similar. This is a fundamental requirement for dynamic similarity. The results of further experiments may be plotted and roughly checked by comparing them with this curve. Any point falling off the curve should be questioned.

[5] "Textbook on Hydraulics", by George E. Russell, M. Am. Soc. C. E., Fourth Edition, p. 433.

It will be noted in Fig. 4 that most of the plotted points fall along a well-defined curve. The equation of this curve is,

$$c_d = \sqrt{\frac{F}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Substituting for F its value as given in Equation (5):

$$c_d = \sqrt{\frac{v^2}{2\,g\,H}} = \frac{v}{\sqrt{2\,g\,H}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

This value of c_d is equal to the coefficient of velocity which indicates that for these gates the coefficient of discharge is equal to the coefficient of velocity, or that the coefficient of contraction is unity. The solution of Equation (8) will give values of c_d for any gate-opening or head, for either the model or the prototype, and for any system of units of measure.

Fig. 4, therefore, might be used to check the consistency of experimental results, to determine the law by which the coefficient of discharge varies, and to predict the discharge of any similar model. In this connection, it would be possible to construct and test a model and then determine the final size of the prototype by ascertaining the scale ratio and scaling up the model, as is done at present for hydraulic turbines.

Probable Errors

The measuring weir was carefully calibrated just before use in this model, and it is believed that the errors resulting from incorrect measurements of discharge are negligible.

There was some fluctuation of head and tail-water elevations in the glass tubes at the model in spite of all attempts to eliminate it. This, probably, caused some error in the experimental results. It is also probable that this error would be compensating.

Possibly there may have been some error in the gate-setting; but this was done with care and the error would be less than 0.02 cm, or $\frac{1}{128}$ in. (0.01 ft, full scale). In constructing the sluices there was some error in width, which ranged from -1.5% to -0.5%, with an average error of -1.0 per cent. This error in the width of the sluice will cause the same percentage error in the discharge.

During the experiment it was suggested to the writer that there might be an error in the zero reading of head that would cause part of the difference between model and prototype results. Accordingly, a series of curves with discharge raised to a power which would produce a straight line as ordinate and gate-opening as abscissa was made for the several heads tested (see Fig. 5). The points used in plotting these curves were obtained from the curves of Fig. 2.

The curves of Fig. 5 indicate that the zero of gate-opening was -0.265 ft for the model experiments and -0.135 ft for the prototype experiments. If the error had been equal for both model and prototype there would have been none in the experimental results. The error in the model, however, is 0.130 ft

greater than that for the prototype. This will cause an average error of +5.1% in the results of the model experiments. Since this error is larger for the smaller gate-openings it would result in a general lowering of the curves of Fig. 3 which would be more pronounced for the smaller gate-openings.

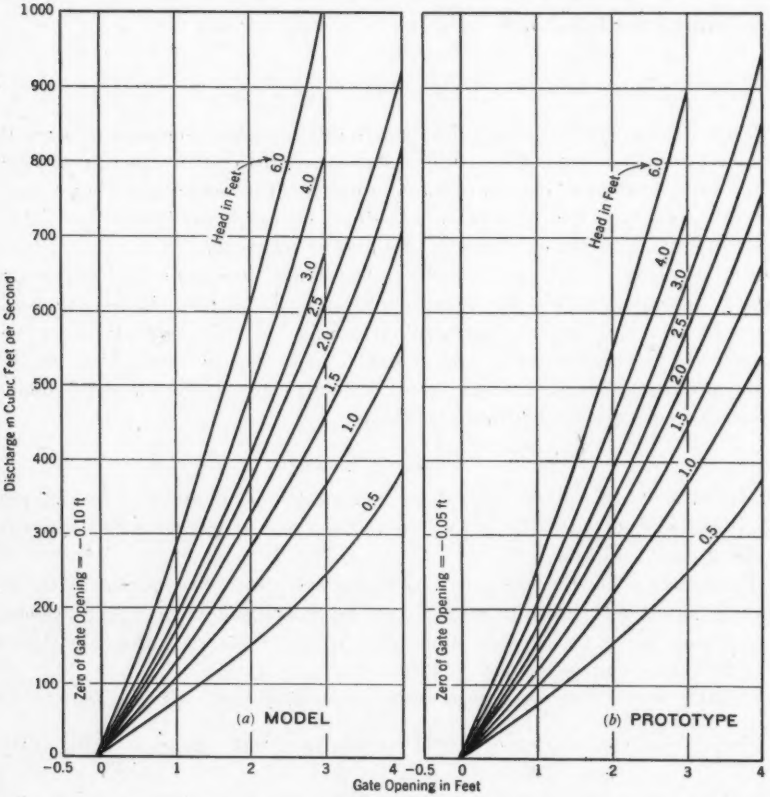

FIG. 5.

In the experiments performed on models of sluices of the Assuan Dam[3] the models were made smooth to the touch (the prototype was of smooth-dressed stone and the values of l ranged from 33 to 67). Subsequently, coatings of fine sand and coarse sand were given one of the models in which l had a value of 33. These sand coatings had the effect of reducing the average discharge by 5% and 9%, respectively, for a fully opened sluice with maximum reductions in discharge of 6.5% and 11%, respectively, the effect increasing with the head. For the sluice four-sevenths open, the water did not touch the top of the sluice, and the average reduction in discharge was 2.4% and 3.1%, respectively, the percentage reduction being nearly constant for all heads. A reduction in discharge of the model will cause positive percentage departures of $\frac{1}{l^{2.5}}\left(\frac{Q_n}{Q_m} - l^{2.5}\right)$, and negative percentage departures of $\frac{C_{dm} - C_{dn}}{C_{dn}}$

Messrs. Hurst and Watt declare[8] that, "* * * if the frictional resistance is not exactly proportional to the square of the velocity, the relation, $Q = q\ n^{2.5}$ [$Q_n = Q_m\ l^{2.5}$] is departed from. * * * The effect of frictional resistance becomes less as velocity and dimensions increase." All these factors are conducive to positive percentage departures from Froude's law. On the basis of the results obtained the writer assumes that the roughness of the model of the Tremont gates simulated the roughness of the prototype fairly well since the latter was fairly rough, the value of l was small, and the model had the roughness of about No. 0 sand paper.

Conclusions

In this paper the writer has attempted to show that: (1) Models of sluice-gates can be depended upon to predict the discharge of their prototypes with reasonable accuracy; (2) Froude's model law will apply to the discharge of sluice-gates; (3) roughness of both model and prototype must be given consideration in the construction of the model; and (4) good results should not be expected for small gate-openings and low velocities.

Acknowledgments

The experiments on the model of the Tremont gates were conducted at the River Hydraulic Laboratory of the Massachusetts Institute of Technology. The paper is based upon a thesis presented to the Institute by the writer in 1934 in partial fulfillment of the requirements for the degree of Master of Science. The complete original manuscript, including the test data, has been filed for research purposes in Engineering Societies Library, New York, N. Y.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

EFFECT OF SECONDARY STRESSES UPON ULTIMATE STRENGTH

Discussion

BY JOHN I. PARCEL, M. AM. SOC. C. E., AND ELDRED B. MURER, JUN. AM. SOC. C. E.

JOHN I. PARCEL,[36] M. AM. SOC. C. E. and ELDRED B. MURER,[37] JUN. AM. SOC. C. E. (by letter).[37a]—The discussions submitted have raised a number of additional points of interest not mentioned in the paper or at most touched upon very lightly, and they have also called attention to certain apparent obscurities of statement. The writers are much gratified that, for the most part, the discussers appear to agree with the main conclusions of the paper.

Mr. Sandberg makes a very good point with reference to the importance of knowing the effect of high secondaries upon the ultimate strength of members in the case of determining the strength of old structures still in service, where it is proper to use a much lower margin of safety than would be the case for a new structure. He mentions the common occurrence of eccentric connections in such structures. It is important to note that secondary stresses arising from this cause are of a character quite different from those contemplated in the paper, as will be further noted in the consideration of Mr. Eremin's discussion.

The point of Mr. Gedo's criticism is not quite clear. He notes as a "disturbing element" that secondary stresses "sometimes are ascribed to the deflection of the truss * * * and sometimes to the gussets". This certainly was not the writers' intention. Secondary stresses as herein considered, of course, arise from a combination of the deflection of the structure and the stiffness of the joints as stated in the first paragraph under the heading, "Analysis of Problem". The frame-work shown by Mr. Gedo in Fig. 22, if the supports are assumed as rigid, would exhibit neither grimary nor secondary stresses.

NOTE.—The paper by John I. Parcel M. Am. Soc. C. E., and Eldred B. Murer, Jun. Am. Soc. C. E., was published in November, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1935, by Messrs. C. H. Sandberg and J. D. Gedo; March, 1935 by Messrs. L. E. Grinter, L. T. Evans, and F. E. Fahy; and August, 1935, by Messrs. LaMotte Grover, and A. A. Eremin.

[36] Cons. Engr. (Sverdrup & Parcel), St. Louis, Mo.

[37] Detailer, Sverdrup & Parcel, St. Louis, Mo.

[37a] Received by the Secretary December 6, 1935.

The first three paragraphs of Professor Grinter's very able discussion contain a particularly clear and forceful statement of the salient points involved in estimating the importance of secondary stress upon ultimate strength. Professor Grinter disagrees with the statement in the paper that the joint rotations upon which the secondary stresses are conditioned increase in direct proportion to the load. He regards this as inconsistent with the title of the paper, which would indicate that the conditions to be considered are those in force when the structure is stressed to its ultimate limit, in which case he states that the displacements would be several times as great as those computed on the basis of elastic behavior.

It is believed that this criticism rests upon a misconception. It is definitely stated in the paper that the effective ultimate strength of a tension member is marked by the yield point of the material, and that ordinarily a compression member will fail at a point slightly above (and, on occasion, at a point slightly below) this stress. The most important conclusion of the paper is that secondary stresses, even when they result in local stress combinations nominally in excess of the yield limit, do not apparently hasten the development of an average stress over the entire section equal to the yield-point stress. So long as the average axial stress remains within the elastic limit, the joint rotations cannot increase faster than the loads, since they are direct functions of the changes of length of the truss members, and the latter, so long as Hooke's law prevails, are directly proportional to the loads. It was not the writers' intention to consider the distortions arising after the average primary unit stresses had reached the yield point.

Professor Grinter apparently feels that one should not place too much dependence upon the analogy between local over-stress due to secondary bending and that due to a number of other causes mentioned in the paper. With this opinion the writers are in entire agreement as should be clear from the statement made in next to the final paragraph of the paper. The point to be made was merely that standard practice at present permits certain violations of the general rule that all stresses should be kept within the yield-point limit, and that the mere fact that secondary stresses exceed this limit does not, without further consideration, establish them as dangerous.

The suggestion that due consideration should be given to the fact that high secondary bending places added strain on the material around rivet holes, which is already strained considerably beyond the surrounding material, is worthy of special consideration. The importance of this effect is considered in some detail in Professor Fahy's discussion. Undoubtedly, further experimental work upon the entire problem of the detailed behavior of riveted and welded joints subjected to a combination of high primary and secondary stresses is much to be desired.

Professor Grinter raises several interesting questions regarding reversal of stress, repetitive stresses, and impact, although the writers are inclined to feel that this interest is largely academic. As regards reversal and repeated stress, the following should be noted:

(1) In any ordinary bridge truss, the only members in which a marked reversal of both primary and secondary stresses takes place are the compara-

tively slender flexible web members near the center (of a simple truss) for which the secondary stresses are very small.

(2) Relatively stocky members which normally exhibit high secondary stresses—chords, end posts, short web members in end panels—ordinarily do not reverse either primary or secondary stress in a truss with simple webbing such as is commonly used for bridge trusses of spans up to about 300 ft. For longer spans, in which sub-paneling is used, some of this group of members may show a very large reversal[38] in live load secondary bending (although not, of course, in primary stress) when the secondaries are computed in the conventional manner. When a more accurate analysis is made, taking into account the induced distortions,[39] the secondary moments are found to be greatly reduced, and, when dead load effect is considered, they will almost never be sufficient to reverse the total extreme fiber stress appreciably.

(3) Structural steel with a yield-point value of 35 000 lb per sq in., may be expected to show a safe "endurance range" of from, say, — 20 000 lb per sq in. to + 35 000 lb per sq in.; or, when no reversal takes place, from 0 to 45 000 lb per sq in. It is believed that the present study shows conclusively that in no case can an extreme fiber stress be developed to exceed the yield-point value (35 000 lb per sq in. in this case) until the primary stress approaches this limit; that is, until failure is impending, regardless of secondary bending. It is also believed that only a "freak" truss will ever show a reversal of total extreme fiber stress ranging from — 20 000 lb per sq in. to + 35 000 lb per sq in. The extraordinary case where such action may take place, of course, will require special treatment.

Professor Grinter argues that impact stresses must be treated separately, since they may be practically instantaneous in character, allowing no time for relief to develop through plastic flow. This is a very important point and requires some further development of the conception of "relief".

If a designer is working to specifications requiring a minimum yield point of, say, 33 000 lb per sq in., and if he finds that in a certain compression member, for example, the computed maximum primary unit stress is 20 000 lb per sq in. and the corresponding secondary stress on the extreme fiber is 18 000 lb per sq in., he will naturally infer that his material at the face of the member is stressed 5 000 lb per sq in. beyond the yield point and, therefore, in danger of collapse due to local buckling. A principal conclusion of the paper was that, due to relief from plastic action, a stress greater than 33 000 lb per sq in. in such a case cannot possibly develop under any degree of secondary bending practically realizable. Remembering that in secondary stress action it is distortions and not stresses that are fixed, it is pertinent to ask: What will be the effect on the material if a given deformation is imposed on a member so rapidly that the material has no time to flow? It may be presumed that in this case a much higher stress will develop, but since it is strain and not stress which, repeatedly applied, tends to break down the material, it is not clear that, as long as the strain is the same, whether imposed instantaneously or gradually, there is any serious effect from the added stress required in the

[38] "Secondary Stresses in the Kenova Bridge", by G. A. Maney and J. I. Parcel, Members, Am. Soc. C. E., Univ. of Minnesota, *Studies in Engineering, No. 4*, p. 29.

[39] *Loc. cit.*, p. 47 *et seq.*

former case. So far as is known there are no experimental data bearing directly on this point.

Professor Grinter states that "it is difficult to visualize impact as producing deflection and joint rotation". It would appear that since such displacements are the direct result of the axial deformations of truss members, as previously noted, they should be as clear and definite as distortions arising from other sources. Any calculation of secondary stresses should always include impact effects as well as those due to static live load, and recent researches appear to indicate that the allowance for impact required in standard railroad bridge specifications is probably considerably greater than is ever actually realized.

Professor Grinter refers to "the authors' suggestion of the entire neglect of secondaries". No such suggestion was made; on the contrary, the writers spcifically state "the results of the investigation do not justify the conclusion that secondary stresses are of no importance". The exceptional cases cited by Professor Grinter as possibilities are among the reasons for this conclusion.

He states further that "in the past the question as to whether secondary stresses should be introduced into a design has been largely academic, because of the lack of a convenient method of secondary stress analysis" and that, "only with the introduction of the Cross method and the writer's simplified method has the designer had available a simple, rapid and understandable method of secondary stress analysis". With this statement, the writers are in complete disagreement. They have the greatest respect for the brilliant method of analysis introduced by Professor Cross and extended by Professor Grinter, and they recognize, of course, that there is always a wide difference of opinion among individuals as to the convenience and effectiveness of different analytical methods. So far as secondary stress computation is concerned, however, an extended experience has convinced them that the slope-deflection method, when the equations are solved by the method of successive substitutions, is a more rapid and convenient method than moment distribution. This method of solving slope-deflection equations has been in use for at least a quarter of a century.

Mr. Evans agrees with the main conclusions of the paper and notes that he has for some time held similar views. In this connection the writers would like to emphasize the statement made in the paper that the idea of large stress relief in the case of high nominal secondaries is by no means new and has been proposed by a number of authorities here and abroad during the past twenty years.

Mr. Evans cites a most interesting case arising in his practice a number of years ago in which a design solution was evolved, which embodied essentially the conclusions of this paper, and the subsequent results of which afford a confirmation of the soundness of the design.

The writers consider Professor Fahy's discussion a most important contribution to the subject. It is so thorough and clearly stated as to offer little room for comment. They have found his detailed study of rivet stresses and comments thereon especially illuminating. As regards the last point made in his discussion—the effect upon gusset-plates of a combination of high secondary and primary stresses—they feel that some further experimental investigation is needed before any final conclusions can be drawn.

Until such information is available, they are inclined to agree with him that in all doubtful cases the edges of gusset-plates should be reinforced.

Mr. Grover raises a number of interesting questions, particularly as regards the effect of secondaries on riveted and welded connections, some of which are partly answered in Professor Fahy's discussion. His questions on the subject of fatigue have been answered in the comment on Professor Grinter's discussion. He refers particularly to the stress concentration arising at the edge of gusset-plates as a point which should receive special consideration in further investigations. If the writers understand correctly what he has in mind, they are inclined to feel that there is no essential difference in the stresses to which he refers and any other secondary stress of equal intensity occurring in the body of a member.

The inward dishing of the cover-plate to which Mr. Grover refers was not offered as a phenomenon which could be depended upon invariably, but rather as one that served to explain in part certain experimental data obtained. Of course, serious rusting or injury causing a marked incipient outward dish in the member would tend to offset the apparent natural tendency of the plate to bend inward.

Mr. Eremin raises two important points which perhaps should have been emphasized more in the paper. It is true that the term, "secondary stress", has been used rather loosely in the literature of structural engineering and is sometimes thought of as any bending stress developing at the ends of a member in a framework with rigid joints. This, of course, is not the type of stress with which the present investigation deals. It is noted in the "Introduction" that secondary stresses are not required to maintain the equilibrium of the structure, which clearly excludes such types as the Vierendeel truss; such structures are unstable without the resistance offered by the rigid joints. Under the heading, "Analysis of Problem", it is stated that,

> "Even when the secondary unit stress is a high percentage of the primary, the secondary moments offer no appreciable assistance in carrying the loads, and the members are always designed on the basis of full hinge action at the ends."

Perhaps this statement is too sweeping a generalization, although the writers know of no case in which a bridge or building truss (using the term, truss, to mean a framework with $m \geq 2n - 3$) has been designed so as to include the secondary bending in the load-carrying capacity of the truss. If a structure were so designed the bending moments in the members would become primary effects and as such are outside the scope of the present treatment.

For the most part, the same remarks apply to so-called secondary stresses arising from joint eccentricity. Such stresses are necessary to maintain the stability of the structure; in general, they will increase substantially in proportion to the primary stresses and, since a certain definite applied moment is to be balanced, no relief will result from local stressing beyond the yield point.

In conclusion, the writers wish to express their thanks to the discussers for their valuable contributions, which undoubtedly add greatly to the usefulness of the paper.

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DISCUSSIONS

RATIONAL DESIGN OF STEEL COLUMNS

Discussion

BY D. H. YOUNG, JUN. AM. SOC. C. E.

D. H. YOUNG,[78] JUN. AM. SOC. C. E. (by letter).[78a]—It is gratifying that the majority of the discussers are in agreement with the proposed basis for the design of steel columns as stated at the end of Section 1 of the paper. However, since some criticism of it has been made on the grounds that it is unjustifiable, it is, perhaps, worth while to recapitulate some of the considerations that led to its proposal.

First, in the case of a compressed column, it is an accepted fact that general imperfections, such as initial crookedness, accidental eccentricity of load, irregularities in cross-sectional area, and lack of homogeneity of the material, affect, considerably, the behavior of the column. It is also well known that the effect of such imperfections can be represented either by assuming a definite initial curvature of the column axis or by assuming a definite eccentricity of the compressive load. Thus, in discussing the behavior of actual columns under load, it is not only justifiable, but absolutely essential, to make one of the foregoing assumptions to allow for the effect of general imperfections, regardless of how indeterminate the proper degree of initial curvature or eccentricity of load may be. Such an assumption is a far more fundamental point than being "merely a method of applying a variable factor of safety" as Mr. Osgood calls it.

Accepting the logic of this assumption, the second step is to consider the general behavior of a pin-ended, initially curved, steel column under the action of a gradually increasing compressive load (see Fig. 4). Such a column begins to deflect laterally from the beginning of loading, and at a certain value of the compressive load yielding begins in the extreme fibers at the cross-section of greatest bending moment. This beginning of yielding may be termed a "localized failure" of the material and the corresponding value

NOTE.—The paper by D. H. Young, Jun. Am. Soc. C. E., was published in December, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1935, by Messrs. William R. Osgood, Alfred S. Niles, J. F. Baker, and K. L. DeBlois; May, 1935, by Marvin A. Gray, Esq.; August 1935, by Messrs. R. G. Sturm and Marshall Holt, F. E. Turneaure, N. J. Durant, E. C. Hartmann, and Edward Godfrey; and December, 1935, by L. T. Wyly, M. Am. Soc. C. E.

[78] Instr., Eng. Mechanics, Univ. of Michigan, Ann Arbor, Mich.

[78a] Received by the Secretary December 2, 1935.

of the compressive load denoted by P_y. As the load continues to increase above the value of P_y it soon reaches a value at which the column buckles; that is, continues to bend laterally, without further increase of load. This condition may be termed "complete failure" of the column, and the corresponding load, P_u, is the ultimate load that the column can carry. In considering whether P_y or P_u should be taken as a basis for the selection of the working load for a steel column, the writer chose P_y for the following reasons:

(1) The beginning of yielding represents a definite damage to the material, if not complete failure of the column, and the load, P_y, which causes such damage may well be regarded as the limit of structurally useful strength of the column. That the Engineering Profession is well agreed on this point is evidenced by the fact that it is accepted practice to select the working loads for beams, subject to transverse loading, on the basis of that loading which first produces yielding in the extreme fibers of the beam.

(2) The determination of the ultimate load, P_u, necessitates a complicated analysis involving the properties of the material beyond the proportional limit (as ordinarily defined). The degree of accuracy with which these properties can be known is not very high and, as a result, calculations of the ultimate load, P_u, are of more academic than practical value.

(3) For most cases of actual steel columns initial imperfections must be expected of such an order of magnitude that there is no great difference between the ultimate load, P_u, and the load P_y, at which yielding first begins in extreme fibers.

When Mr. Osgood criticizes the proposed basis for design on the grounds that "it results in excessive safety for short columns, * * *, and it makes the H-section appear stronger than the rectangular section, * * *", he is using as a definition of strength the ultimate buckling load, P_u. The writer believes that, to be consistent, Mr. Osgood should object also to the present, well accepted, basis for the design of steel beams, subjected to transverse loading. In this case, also, current design methods make the I-section appear stronger than the rectangular section when judged on a basis of ultimate strength. In fact, this difference is even greater in the case of beams than in the case of columns.

Finally, in calculating the load, P_y, the writer assumes the material to follow Hooke's law up to the yield-point stress. In his discussion, Mr. Osgood concludes that "it is entirely unjustifiable, therefore, to assume the ideal stress-strain diagram of Fig. 2 in explaining the behavior of columns, except possibly at large values of the eccentricity ratio." As a matter of fact, there is good justification for this assumption on two counts:

(1) The writer concludes from Fig. 8 that the "large values of the eccentricity ratio", to which Mr. Osgood refers, are probably never greater than $\frac{\Delta}{k} = 0.1$. Indeterminate as the factor, Δ, may be, there is good indication that (except, possibly, for very short columns, $\frac{L}{r} < 40$, which always fail

at $\frac{P}{A} = f_y$) general imperfections necessitating an allowance for $\frac{\Delta}{k} > 0.1$ must be expected in the case of actual columns.

(2) As pointed out by Professor Turneaure, in his discussion, "the variation in yield point and in the form of the stress-strain curve in the vicinity of this point, * * * renders a more exact treatment * * * of little practical value for structural columns."

Development of Rational Design Formulas.—Following the proposed basis, rational design formulas are developed in the paper for the case of a pin-ended column under two conditions of loading: (1) the condition of loading considered in Section 2 of the paper, which is simply that of two compressive end loads; and (2) the condition of loading, considered in the first part of Section 4 of the paper, which is that of compressive end loads applied with given (not accidental) eccentricities, e_0 and e_1. For Condition (1) it is most convenient to allow for the effects of general imperfections by assuming an initial shape of the column axis in the form of a half-sine wave having a maximum deviation from a straight line through the ends of Δ. The resulting formula for determining the loading which first produces yielding is represented by Equation (12). For Condition (2) it is most convenient to allow for the effect of general imperfections by increasing the given eccentricities, e_0 and e_1, algebraically by equal amounts, e. The resulting formulas for determining the loading which first produces yielding are represented by Equation (19) and Equation (22). As mentioned by Professor Niles, design formulas can be derived by the same procedure for other conditions of loading, for example, such as a combination of axial and transverse loads.

Considerable discussion hinged upon the writer's choice of $\Delta = \frac{L}{400}$, to allow for the effect of general imperfections in the fundamental case of a pin-ended column. One finds, among the discussions, suggested values ranging from $\frac{L}{1\,200}$, as given by Mr. Gray, to $\frac{L}{300}$ as given by Mr. Godfrey. The writer chose $\frac{L}{400}$ from a study of recommendations given by various experimental investigators. Professor Turneaure suggests that this value seems too severe and points out that, as a result of its studies, the Special Committee of the Society on Steel Column Research adopted a crookedness value which corresponds approximately to $\Delta = \frac{L}{1\,200}$. It must be kept in mind, however, that the value, $\Delta = \frac{L}{400}$, as chosen by the writer, is intended to allow, not only for crookedness, but also for accidental eccentricity of load, variations in cross-sectional area, and all imperfections in general. For the most part, the Committee used special bearing-blocks by which eccentricity of load could be controlled and, hence, no conclusions can be made from such tests, regarding the possible extent of accidental eccentricities of load in the case

of columns in actual structures. The writer agrees with Mr. Wyly that a careful study conducted, not only in the laboratory, but also in the field and shop, is necessary for determining the proper allowance for the effect of general imperfections. Mr. Wyly's discussion shows clearly the variety of factors which must be considered in the adoption of a reasonable value for such a factor as Δ. Particularly does it show that initial crookedness is by no means the predominate factor among general imperfections and that errors in alignment, etc., may result in very high accidental eccentricities of load. Thus, the writer does not believe that his proposal of $\Delta = \frac{L}{400}$ is by any means ultra-conservative. Mr. Wyly, however, presents good evidence to the effect that the value taken for Δ should include a constant term plus a term that increases with the length, L.

Rational Formulas vs. Empirical Formulas.—Equation (12) will serve as a basis for a comparison between rational and empirical design formulas of the straight-line and parabolic types. The principal advantage of the rational formula is that it shows explicitly what factors affect the strength of a column and, in addition, just what effect each factor has. For example, Equation (12) shows the strength of a pin-ended steel column to depend upon the following factors: (*a*) Properties of the material, as represented by f_y and E; (*b*) the extent of general imperfections, as represented by Δ; (*c*) the slenderness of the column, $\frac{L}{r}$; and (*d*) the shape of the cross-section, as defined by the core radius, k, as compared with the radius of gyration, r.

The only argument ever advanced against the rational formula is that it is cumbersome to use in practice. Regarding empirical design formulas, it is beyond question that the only argument in their favor is their simplicity.

As to the arguments against them the writer can do no better than to refer to the discussion by Mr. Durant, with which he is in complete agreement. To Mr. Durant's arguments against the empirical formula, one more may be worthy of addition; namely, that there is a definite psychological advantage in having the designer work directly with a rational formula which, like Equation (12), for example, keeps him constantly reminded of the various factors that influence the strength of the column with which he is dealing. In this way he may be able to take advantage, whenever possible, of favorable conditions, regarding one or another of the aforementioned factors and thus make some saving in material or improvement in design. On the other hand, if he works day after day with a straight-line or parabolic type of formula in which $\frac{L}{r}$ appears to be the only factor affecting the strength of a column, he may soon forget the general characteristics of column action and, particularly, the inherent uncertainties which go with it.

Columns with Other Than Pinned Ends.—Regarding the adaptation of Equation (12) to the case of a column with other than pin-ended conditions of support, the writer agrees with Messrs. Sturm and Holt and with Mr.

Hartmann that this can be done by using in Equation (12) a modified value of s_e, as given by Equation (86). This is an important point since it extends considerably the range of application of Equation (12). Furthermore, it is a point that was not stressed in the paper, and it is well that these discussions have called attention to it. However, as a general method of dealing with columns having other than pin-ended conditions of support it has its definite limitations and since these limitations were not mentioned by the discussers it is, perhaps, worth while to call attention to them.

To begin with, such a procedure evidently takes for granted that when yielding first begins in the extreme fibers, the column axis will have such inflection points that the "free length" between them will be equal to the reduced length, $L' = \frac{L}{N}$, as determined by Equation (86). In general, this circumstance will be realized only by virtue of the existence of two conditions. The first is that, to allow for the effect of general imperfections, one assumes an initial shape of the axis of the column which agrees in form with the configuration of elastic buckling consistent with the constraints. Only by virtue of such an initial shape, will the "free length" between inflection points remain constant during loading and equal to the effective length, $L' = \frac{L}{N}$. In view of the highly indeterminate nature of general imperfections, it seems justifiable in any case to assume such an initial shape, thus fulfilling the first condition in effect if not in fact. The second condition which must be realized is that, during loading, any moments arising at points of constraint must be independent of change in length of the column due to compression, because, regardless of the choice of an initial shape, if such secondary moments do arise they will cause the "free length" between inflection points to change gradually during loading. For such cases it is not justifiable to use a reduced length, $L' = \frac{L}{N}$, in Equation (12), in calculating the load that first produces yielding.

In summary, the use in Equation (12) of a modified value of s_e, as given by Equation (86), for the calculation of the load first producing yielding in the case of a column with other than pin-ended supports will be justifiable: (*a*) For all cases of columns in which the manner of support is such that the constraints are of an inelastic nature (see Fig. 31); and (*b*) for columns with elastic constraints, providing the arrangement is such that bending of the column, during loading, is independent of any change in its length due to compression (see Fig. 32). The same procedure will be unjustifiable for columns with elastic constraints where the arrangement is such that bending of the column, during loading, depends upon change in its length due to compression (see Fig. 33). Cases such as those shown in Fig. 33 represent column action in rigid-frame construction and require special consideration.

The writer does not agree with Professor Turneaure that any allowance in the form of a reduced length equal to 85% of the true length should be made for possible end constraint due to friction in the case of pin-ended

columns. Such a procedure might be justifiable if all loads on a pin-connected structure were static, but in the case of bridges, subjected to moving loads, vibrations of the structure are certain to be set up and such vibrations

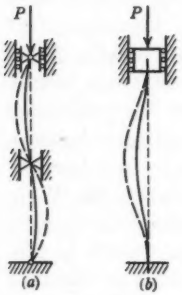

FIG. 31.

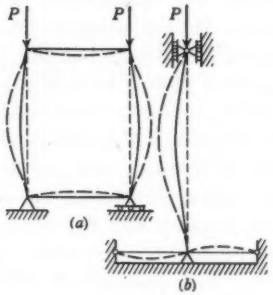

FIG. 32.

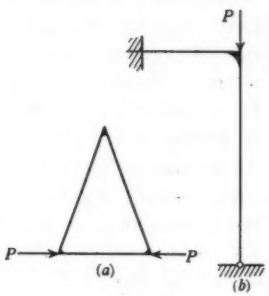

FIG. 33.

can easily result in a gradual rotational creep of the ends of a pin-ended compression member. Thus, it seems that frictional constraint at the ends of a pin-ended column is too uncertain a factor to justify any advantage being taken of it and that under certain conditions a pin-ended column designed on the basis of Equation (94) may by no means have the intended factor of safety.

Columns in Rigid Frame Construction.—It is pointed out in the paper that an exact analysis of column action in rigid-frame construction is too laborious to be practicable. Instead, it is proposed to calculate all secondary end moments by the usual approximate method and then to use Equation (19) and Equation (22), or curves, such as those of Fig. 12, plotted from these equations, as a basis for design. Such a procedure amounts to the assumption that the effective length of the member is the same as its true length and also that the secondary end moments increase in direct proportion to the axial compressive force in the member. Both these assumptions are conservative and hence the use of Equations (19) and (22), or curves like those of Fig. 12 plotted from them must, of course, result in some extra safety. However, in many practical cases, this indeterminate degree of extra safety is by no means always as large as might be expected. For example, in the case of a truss, the chord members are usually much more rigid than the web members. As a result of this the effective length of a web member will be considerably less than its true length. On the other hand, web members are usually bent in double curvature so that the maximum fiber stress occurs at one end and the member can be designed on the basis of Equation (19) where no question of effective length is involved. Again, as emphasized by Mr. Durant, chord members receive little constraint from the web members, and their effective lengths are approximately their geometric lengths. This is particularly true when one considers the behavior of a chord member in a plane perpendicular to the plane of the truss. Thus, in such cases, there is little to prevent bending of the chord members in alternate directions in successive panels. This raises the question as to whether

or not the Special Committee on Steel Column Research is justified in its selection of an effective length equal to 75% of the geometric length for all cases of compression members in a riveted truss.

Built-Up Columns.—Regarding the strength of a built-up column as a unit (that is, assuming adequate lacing, diaphragms, etc.), Equation (12) can be used as a basis for design by introducing the fictitious length, L', as given by Equation (27) or Equation (28). In this case, the fictitious length, L', is greater than the true length, L, thus compensating for the additional flexibility of the column due to shearing deformation of the lacing or battens.

Regarding the proportioning of details, such as lacing or batten bars, to make certain that the built-up column will act as an integral unit, Mr. Wyly raises considerable objection to present specifications regarding this point. The writer believes that a proper basis for the design of such details is presented in Part II of the paper. The manner of allowing for the effect of general imperfections, as regards shear, is discussed in the second half of Section 5, and corresponding design formulas (Equations (33) and (39)) are developed in Sections 6 and 7. The manner of allowing for shear due to secondary end moments which occur in the case of a column in a truss, is discussed fully in Section 8, and Equation (45) is proposed as a basis for design. It is seen that in the case of a short column under the most unfavorable conditions, both as to general imperfections and secondary end moments, the necessary allowance for shear may be so high as to prohibit the use of a column of non-solid cross-section.

Mr. Durant points out that to be perfectly rigorous in the derivation of the equations of Sections 6, 7, and 8 of the paper, the writer should have used as the basic differential equation of the elastic curve, Equation (97), instead of Equation (96). Although this is true the writer believes that the error in the average shear stress, v_y, as given by Equations (33), (39), and (45), can be somewhat compensated for by using, in these equations, or in the curves plotted from them, the fictitious length, L', as given by Equation (27) or by Equation (28). Incidentally, the error appears to be on the side of safety.

Miscellaneous.—Messrs. Sturm and Holt attempt to extend the range of application of Equation (12) to the determination of the inelastic buckling load by the use of a reduced Eulerian value, s', as defined by their Equations (83), (84), and (85). However, they give no justification for such a procedure, except that values of s_u, calculated from their Equations (87) and (88) agreed "satisfactorily" with certain test results[47]. The principal obstacle to the use, in Equation (12), of a reduced Eulerian value, s'_e, calculated by the use of a reduced tangent modulus, is that such a substitution (see Equation (87)) amounts to the assumption that the material throughout the full length of the column behaves in accordance with this reduced modulus, whereas this inelastic behavior is usually confined to a relatively short portion of the length of the column.

Mr. Hartmann proposes a column formula (Equation (108)), the use of which he advocates in actual design. Mr. Hartmann's exceedingly careful

[47] "Strength of Tubing Under Combined Axial and Transverse Loading", by L. B. Tuckerman, S. N. Petrenko, and C. D. Johnson, Technical Notes of National Advisory Comm. for Aeronautics, No. 307, June, 1929.

definition of the factor of safety, n, scarcely seems consistent with the manner in which, subsequently, he illustrated the application of this formula. The writer's equations for allowance for shear in a pin-ended column and the 1925 A. E. R. A. Specifications. He assumes that the factor of safety has

Professor Niles shows how the equations in Sections 4 and 8 can be simplified somewhat for practical use by retaining in them the variable, x_m, which defines the position of the section of maximum bending moment. His Equations (54) to (58) should prove useful to any one who desires to plot sets of curves like those of Figs. 12 and 18.

Mr. Hartmann questions the comparison made in Fig. 16 between the writer's equations for allowance for shear in a pin-ended column and the 1925 A. R. E. A. Specification. He assumes that the factor of safety has been overlooked and suggests that values of $\frac{v_y}{n}$ instead of v_y should have been plotted, which would have had the effect of lowering all the writer's curves considerably. As a matter of fact, what the writer did was to double all ordinates of the straight line representing the A. R. E. A. Specification on the assumption that it intended a factor of safety of approximately two. This surely is equivalent to Mr. Hartmann's suggestion.

In closing, it is worthy of notice, as brought out by Professor Baker, that a rational column formula, including the effect of all factors in Equation (12), except that of the shape of the cross-section, has been incorporated in the British Standard Specification for the Use of Structural Steel in Buildings and that a British Steel Structures Research Committee is considering the problem of steel columns subjected to secondary end moments, as in the case of compression members in rigid frame construction. It is hoped that this paper may have some influence in bringing the design of steel columns in the United States to a more rational basis than is represented by present-day practice. The writer agrees completely with Mr. Gray in his statement that the paper "touches only lightly on the subject", but he agrees also with Mr. Godfrey in that "its attack is in the right direction."

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ELASTIC PROPERTIES OF RIVETED CONNECTIONS

Discussion

BY MESSRS. JOHN SANFORD PECK, AND J. CHARLES RATHBUN

JOHN SANFORD PECK,[22] ASSOC. M. AM. SOC. C. E. (by letter).[22a]—Under "Description of Tests and Failures" (see following Fig. 11) the author states: "The curve obtained was of an entirely different type from that of the others in this entire series of tests. This curve suggests a subject for further investigation." He was referring to Specimen 14, Run 3, in which a specimen that had been tested beyond the elastic limit, but not to destruction, was loaded in the reverse position, thus simulating the condition arising in a wind-bracing connection when the direction of the wind changed 180° (see Fig. 3(*b*), Specimen 14, Run 3.). A further inquiry into this condition, therefore, is justifiable. Two types of connections were fabricated for testing in the laboratory of the College of the City of New York, identical with Specimens 13

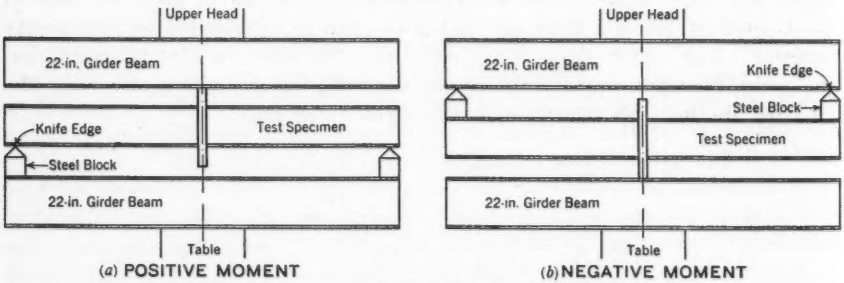

FIG. 45.—METHOD OF LOADING.

and 14 (see Fig. 1), except that the over-all length was 10 ft instead of 6 ft. Two specimens of each type were made up, the entire investigation thus comprising tests of two pairs of specimens.

NOTE.—The paper by J. Charles Rathbun, M. Am. Soc. C. E., was published in January, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1935, by Ralph E. Goodwin, Assoc. M. Am. Soc. C. E.; May, 1935, by Messrs. Harold C. Rowan, Walter Scholtz, J. F. Baker, L. E. Grinter, and C. R. Young and K. B. Jackson; August, 1935, by E. Mirabelli, M. Am. Soc. C. E.; and November, 1935, by R. L. Moore, Esq.

[22] Asst. Prof., Civ. Eng., Coll. of the City of New York, New York, N. Y.

[22a] Received by the Secretary December 2, 1935.

The experimental technique was identical with that followed by the author. The dials used in measuring the angular rotation were placed as shown in Fig. 6, except that Dials *M*, *N*, *O*, and *P*, were omitted. The method used to reverse the direction of the application of the load is shown in Fig. 45 the knife-edges, *AA*, being changed from the bottom to the top position when it was desired to change the direction of loading.

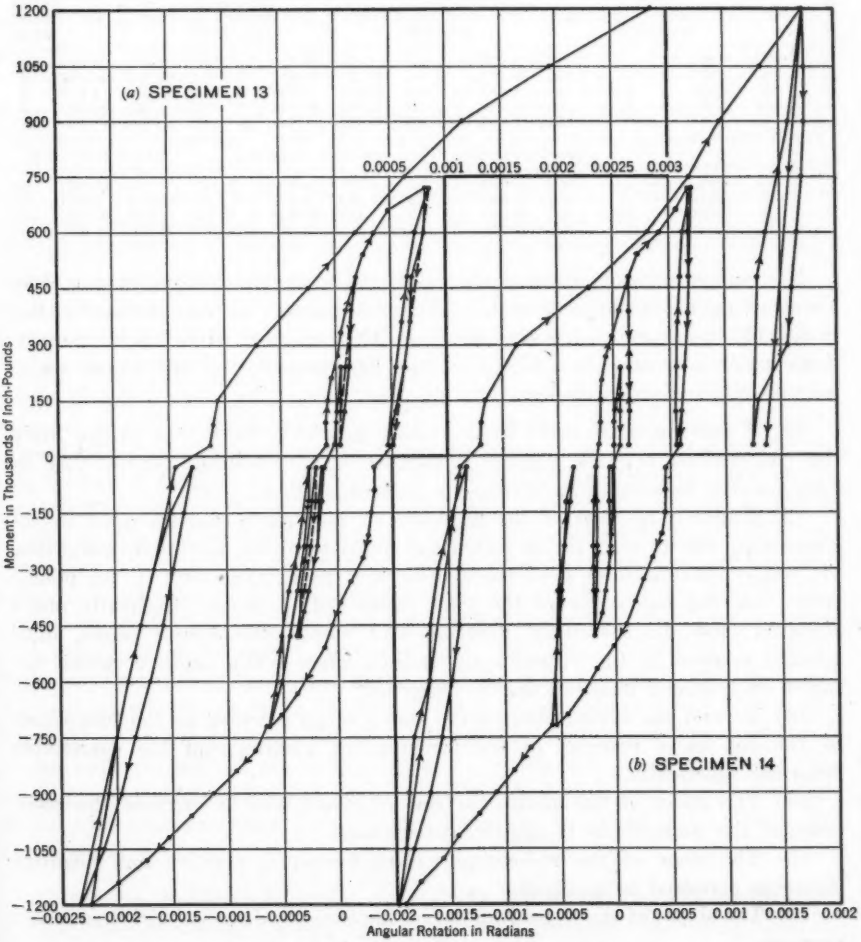

FIG. 46.—RESULTS OF TESTS ON SPECIMENS.

The specimens were loaded according to the schedule given in Table 8, the loads being the total applied to the central plate of the specimens. In Figs. 46(*a*) and 46(*b*), the angular rotation, expressed in radians, of the connection was computed by the same method as that used by the author, and the results of each side were averaged. The results from the tests of the two identical specimens

were then averaged so that the data shown, represent the average of tests of four connections, respectively.

TABLE 8.—SEQUENCE OF LOADING

Loading cycle	Load, in Pounds		Successive increments, in thousands of pounds	Loading cycle	Load, in Pounds		Successive increments, in thousands of pounds	Loading cycle	Load, in Pounds		Successive increments, in thousands of pounds	Loading cycle	Load, in Pounds		Successive increments, in thousands of pounds
	From:	To:			From:	To:			From:	To:			From:	To:	
1	1 000	8 000	2	9	− 1 000	− 8 000	− 8	18	1 000	16 000	16	27	− 1 000	−24 000	−24
2	8 000	1 000	4	10	− 8 000	−16 000	− 2	19	16 000	24 000	2	28	−24 000	−40 000	− 2
3	1 000	8 000	4	11	−16 000	− 1 000	− 4	20	24 000	1 000	4	29	−40 000	− 1 000	− 5
4	8 000	1 000	8	12	− 1 000	−16 000	− 4	21	1 000	24 000	4	30	− 1 000	−40 000	− 5
	DIRECTION REVERSED			13	−16 000	− 1 000	−16	22	20 000	1 000	20	31	−40 000	− 1 000	−40
					DIRECTION REVERSED				DIRECTION REVERSED				DIRECTION REVERSED		
5	− 1 000	− 8 000	− 2	14	1 000	16 000	2	23	− 1 000	−24 000	− 2	32	1 000	40 000	5
6	− 8 000	− 1 000	− 4	15	16 000	1 000	4	24	−24 000	− 1 000	− 4	33	40 000	1 000	5
7	− 1 000	− 1 000	− 4	16	1 000	16 000	4	25	− 1 000	−24 000	− 4	34	1 000	40 000	10
8	− 8 000	− 1 000	− 8	17	16 000	1 000	16	26	−24 000	− 1 000	−24	35	40 000	1 000	40

The writer presents these curves with the idea that the reader can draw his own conclusions from the data. The small number of tests involved makes it difficult to generalize too extensively on the action of wind-bracing connections under a reversal in wind direction. The tests do indicate trends which may be summarized, as follows:

(1) Under repeated loads in the same direction within the elastic limit, the connections behaved elastically because the re-loading curves may be considered to be straight, allowing for hysteresis effect.

(2) Under a reversal of the direction of loading, a slip occurred in the connection, due to some factor which was not determined in this investigation. (It might possibly have been the slackness of fit of the rivets in the holes.) After this slip had occurred the joint continued to behave elastically under repeated loads in the same direction and within the elastic range, until another reversal in the direction of loading, when a slip again occurred followed by a second phase of elastic behavior.

(3) Beyond the elastic limit there was a large increase in the magnitude of the slip on a reversal in the direction of loading and the connection deformed plastically.

(4) The effect of the additional row of rivets was to increase the resistance of the connections to angular deformation.

(5) The slope of the re-loading curves for equal positive and negative moments appeared to be equal.

(6) The slopes of the re-loading curves appeared to decrease with increasing moments.

Direct comparison of the data presented herein with those presented by the author is difficult for the reason that the conditions under which the two sets of tests were made were not identical. The use of a longer span decreased the effective shear on the connections as tested by the writer and, although it may be argued from theoretical consideration that the shear on the joint has no effect on the angular rotation, there is not sufficient experimental evi-

dence supporting that point of view to warrant the comparison of two sets of data having different conditions of shear acting on the connections.

The writer's tests, however, do serve to accentuate a point which the author has failed to emphasize particularly. Figs. 46(*a*) and 46(*b*) show clearly that the slope of the reloading curve is not a constant quantity, but is dependent on the degree to which the joint has previously been strained. The author has chosen his coefficient, Z, such that $M Z = \theta$, in which θ is the angular rotation of the joint for a given moment, M. From the foregoing equation, $Z = \frac{\theta}{M}$, in other words, Z is the reciprocal of the slope of the re-loading curve. From an inspection of Figs. 46(*a*) and 46(*b*) it is readily seen that for a given moment there can be several values of Z depending upon the previous history of the connection. This is not clearly brought out by the author either in his curves or in his test. For one who is familiar with the interpretation of experimental data and the general technique of re-loading curves, the information is available with which to make that deduction, but it is doubtful whether the average reader is familiar enough with laboratory technique to be able to read that most important fact for himself from the data as presented, and it is to be regretted that the author did not bring the point out more explicitly. In view of the fact that Z is a variable coefficient that differs with the size and type of the joint and with the degree to which the joint has previously been strained, it would seem that the use of a numerical value for Z in actual steel design must be based upon the completion of a long series of laboratory investigations undertaken to determine its exact numerical values.

The author has undoubtedly made a major contribution to steel design in developing his theory of the elastic behavior of riveted connections, but the writer foresees a long period of necessary laboratory work running into several years before this theory can be used in practical designing. At present, those connections which developed relatively small resistance to moment (namely, Series *A*, Specimens 1 to 7), are taken as being free end beams whereas those connections which developed large resistance to moment (Series *B* and *C*, Specimens 8 to 14) are designed as completed fixed-end connections. The use of the author's method of design, therefore, would not affect the design of the clip-angle connections and would increase the amount of steel necessary in the wind-bracing connections; hence, the result would be less economy of steel than at present.

J. CHARLES RATHBUN,[23] M. AM. SOC. C. E. (by letter).[23a]—The discussions of this paper emphasize the fact that the problem presented was being studied in at least three other laboratories at the same time that the work was being conducted by the writer, indicating that the subject of stress analysis in steel frames has reached a stage where the properties of the connections should be considered. It is fortunate that the synopsis and reference of each of the works of the other investigators have thus been published in order that those

[23] Associate Prof., Civ. Eng., Coll. of the City of New York, New York, N. Y.
[23a] Received by the Secretary December 2, 1935.

who are interested further in this subject may read the originals if they choose. The writer undertook the experiments described in his paper because he was unable to find any data on the subject. Certain phases of the experiments touched upon by the discussers (such as the reversal of moment, the distribution of deformation among the parts of the connection, and the effect of shear) were omitted because they were not needed for the solution of the problem of which this investigation was a part.

Professor Goodwin has taken advantage of his familiarity with the writer's work and methods to clarify the paper in some respects, and his comments are appreciated. The question of signs is one that arises quite often in studies of this nature, partly because textbooks are often confusing on the subject. The system used by the writer for beams is to accept the conventional sign for moment on a beam as positive when the upper fibers are in compression. As E and I are always positive, the signs for shear, load, slope, and deflection follow at once by direct integration or differentiation of the mathematical expression for moment. Some confusion has arisen because the foregoing statement leads to negative loads when their direction of action is negative (downward) and to negative deflection when this deflection is downward. The desire to avoid these two situations is probably the reason for using other sign systems in computations. The convention for the signs of the moment tending to rotate a joint is used as positive when counterclockwise for the reason that counterclockwise angles, when small, have positive tangents. Thus, stress is proportional to strain and the signs are consistent in this relation. Those who wish to consider clockwise motion as positive will find no error in their work.

The beam and joint sign conventions cannot be consistent with each other at all times. In the case of a beam a counterclockwise moment is positive at one end and negative at the other. The writer knows of no better method of minimizing this difficulty than the introduction of two self-consistent sign systems. As Professor Goodwin states if the reader is familiar with another sign system than that used by the writer the formulas are developed in the paper and it is a very simple matter to go over them changing the signs to the more familiar system. The writer feels that Professor Goodwin has done a service in clarifying this point.

Mr. Rowan and Professor Baker infer that the writer should have mentioned the work of Professor Batho[10] in his "Introduction". This omission was unintentional and was due to the fact that the paper was submitted before Professor Batho's report was available. The writer is in accord with Professor Baker that those who are working on any particular problem should be in closer touch with each other. Discussion reveals that the British Steel Works Association has been working on this subject for several years without the knowledge of those committees of the American Institute of Steel Construction and of this Society as well as of Columbia University, which were consulted during the investigation, or examined the writer's paper before it

[10] Second Rept. of the Steel Structures Research Comm. of the Dept. of Scientific and Industrial Research, pub. by His Majesty's Stationery Office, London, England, 1934, p. 92.

was published. At the same time, these organizations have been familiar with the plans of the investigation since 1931, and the tests have been advertised locally as they were conducted, so that those who were interested might view them and suggest ideas. It is unfortunate that practical difficulties prevent a closer co-operation in the advancement of the field in which several investigators are interested.

Professor Batho's work is similar in many ways to that of the writer. It contains many ideas not brought out in the paper or discussions. The duplication of the several tests is not an economic loss as a study of the additional number of tests lends weight to the value of the constants used in computations.

The remarks concerning the only possibility of connection failure being due to fatigue are not correct. A frame that consists of columns and beams under horizontal loads relies upon the connections for its strength unless they are stronger than the beam itself. It is true that failures of this type do not occur in buildings, probably because horizontal loads are not applied to buildings with as much force as would be necessary to cause the trouble suggested. The problem is of more than academic interest, however. One method of bracing for horizontal loads is to use wind-brace connections of the type of Series *C*, because such connections have a high resistance to moment.

Mr. Rowan's suggestion that column sections introduced between the connections would produce results of a different character is undoubtedly true. As there are so many combinations of column section and connections, the writer is of the opinion that the properties of columns as affecting the rigidity of joints can be made a separate study. The combination of connection and column can then be developed, as can the combination of several connections. The writer tested only one column, because his paper deals with connections. As Professor Grinter has indicated, some idea of the effect of columns can be obtained from this test. The field is full of subjects for investigation but, to be of value, a paper must confine itself to the subject under discussion. The subject of shear, therefore, was omitted from the paper, perhaps unwisely. The question of partitions and concrete in buildings is also entirely out of the province of the subject.

One reason shear has not been discussed is that the data of the tests have a wide spread. To conduct a series of tests varying the shear would involve considerable labor and might lead to a minor correction in the results which by their nature are erratic. If one is interested, this is a subject for separate investigation, but the writer feels that he has made his paper of sufficient length within its well-defined scope.

Shear angles were omitted on the connections of Series *C* in order to observe the effects of such omission. A study of the shear curves will give designing engineers the data from which they can decide whether or not such angles are necessary.

Mr. Rowan's remarks about "complete analysis" and "trial-and-error method" are not consistent with modern progress. Twenty-five years ago some engineers might have agreed, if they were strong opponents of the prog-

ress being made in stress analysis. As to the "trial-and-error" philosophy, the writer respectfully refers Mr. Rowan to the published opinions of Hardy Cross, M. Am. Soc. C. E.

Mr. Scholtz is to be congratulated in indicating how the labor of allowing for the connections can be reduced in an analysis. The writer feels that the improvements in this investigation that are to be made are in theory rather than in laboratory tests. Although more tests are always of value the digesting of these tests and the incorporating of their results in engineering design is another and more important matter.

Professor Batho's graphical method of analyzing connections is one that deserves a more complete treatment than the outline given in the discussion. It should help materially in popularizing the idea of including connection data in a frame analysis.

Both Professors Baker and Grinter suggest that some of the terms in Equations (15) to (21) could be omitted to advantage. This is the intent of the writer. Having given the several terms that theoretically affect the results, Table 3 is constructed so as to include these terms in the computations. If one does not wish to use the corrections, the consequent error can be approximated after a study of the table.

The writer is inclined to agree with Professor Grinter that in the practical case the moment-rotation curves may be considered as straight lines. In fact, the reloading curves are not far from straight. The experimental data indicate that after a connection has received some stress the reloading curves govern. This should tend to simplify the computations. The slope of the curve must be based on judgment after the probable history of the connection has been considered.

As the tests described by Professors Young and Jackson were conducted with objectives differing from those of the writer, a thorough study of their work should reveal many ideas not treated by the writer. The suggestion that the writer neglected the study of shear, for example, is mentioned by several of the discussers. To have made an investigation along these lines would have multiplied the number of tests, the number of specimens, the time, the labor, and the expense very much. The effect of shear, the writer feels, is small compared with the variation in the properties of connections of the same design. Consequently, he has left it for others to study.

Although the laboratory technique used by the writer differed from that of the several discussers, he feels that it was as satisfactory as the others. It was adopted after considerable study of the idea of fastening the instruments beyond the influence of the connection. No trouble was experienced with the dials slipping on the bands, due to shear deformation. The data were not recorded, nor the curves plotted to a degree of accuracy comparable to readings of 0.0001 in., and the writer does not wish to create the impression that the work was done with that refinement. Dials of this degree of accuracy were used because they were easy to read and were readily available.

The question of reversal of moments was considered while planning the series of tests made by the writer. This aspect of the subject was not studied for several reasons: One was that the paper had already assumed an excessive

length; another was that an investigation of the deflection of one of the tall buildings in the City of New York (which deflected less than 6 in. in eighty stories during a 90-mile wind) led to the impression that the wind would not reverse the dead moment on a building connection under any but the most exceptional circumstances. The writer felt that he may have been in error in neglecting this phase of the problem and, therefore, a study of reversals was undertaken at the College of the City of New York as soon as the eighteen tests made by the writer were completed. These tests were made by Professor Peck who has presented his findings in a separate discussion.

The question of whether the top or bottom attachment is responsible for the largest percentage of deformation was not studied and was not treated in the paper although the necessary data were taken during the tests. This phase of the problem did not have a bearing on the work that the writer had in mind. As illustrated in the curves, the right half and the left half of the specimen each constituted a separate test of a connection. These tests were both plotted and published. This was also done in the case of reversed loading, but in the latter case, only the average curve is published, for the sake of clearness.

Professor Mirabelli demonstrates, by means of curves, the effect of the change in the properties of the connections as the load is increased, with the conclusion that the effect is less as the load increases. This is a material contribution to the paper. His statement that the writer's percentages are based on rigid columns is true; the necessary changes can be made when the design is known. This makes the data different from those of the problem investigated by the writer, with the result that different answers will be obtained.

The third point indicates that the transfer of the moment from the beam to the column will increase the size of the column. The fact is that the moment is in the column whether it is computed or not. It follows, therefore, that it is in the interest of economy and safety to evaluate this moment as nearly as possible when designing columns. This is an argument against the present practice of designing beams with connections considered as having a rigidity of either infinity or zero.

It is true, as stated, that the more accurate analysis is the more expensive and that the approximate methods will be used in most commercial work. The question that must be solved in each individual case is whether the more correct solution is worth the expense involved.

Mr. Moore has given an entirely different approach to the problem in which he shows that similar investigations are being conducted in aluminum. He introduces the degree of fixity, or the ratio of the bending moment developed in the spliced beam, over the support, to that obtained in a continuous beam. To obtain this ratio he observed the stresses at about eight sections of the beam and computed back to the moment at the connection. This avoided the errors that might have arisen from measuring directly at the connection itself. It is unfortunate that the mathematical equations resulting from this method of attack were not given so that they might be compared with those in the paper, the best in each system being retained.

The tests made by Professor Peck were designed to obtain curves that can be compared directly with those of the paper, furnishing information on the effect of reversal of moment on the connections. The only difference between the tests of Specimens Nos. 13 and 14 and those offered by Professor Peck is the ratio between shear and moment. The reason for the change in length of the specimens from those previously used was that the early tests indicated serious deformation in shear before the moment curves had been developed as much as was desired. Although the shear may have some effect on the action of the connection, the writer believes that this question can be approached from a theoretical angle as was done by Professor Batho, but an experimental approach would be very expensive and difficult.

The decrease in the slope of the loading curve, when the connection receives high stresses, is of interest. This is only one of the properties of connections that must be considered when designing a structure. The history of the connections is of necessity a part of the problem of stress analysis. This history is not difficult to anticipate when the frame with its anticipated loads is given.

Tests of the type discussed in the paper are, by their nature, expensive and, therefore, the number cannot be great for financial reasons, although tests are always of value. The writer does not agree with Professor Peck that a long series of laboratory investigations are necessary before the theory can be used in practical design. The question is entirely an economic one. Sufficient tests have been made so that a general idea of the constants of the connections can be anticipated over the range covered by these tests and even beyond.

It is hoped that the properties of riveted connections will receive further study from the profession because, undoubtedly, they affect the distribution of stress. It is the writer's opinion that the greatest progress can be made by perfecting the method of analysis rather than in increasing the data by laboratory tests. When a method of analysis that is simple and easily applied is devised, it will be accepted by designers. This does not mean that further tests are not of great value but rather that the value of further research in the theory is greater.

In closing, the writer wishes to thank those who have contributed from their fund of information to this important subject.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

STABILIZING CONSTRUCTED MASONRY DAMS BY MEANS OF CEMENT INJECTIONS

Discussion

BY D. W. COLE, M. AM. SOC. C. E.

D. W. COLE.[10] M. AM. SOC. C. E. (by letter).[10a]—In the several discussions of this paper there has appeared the recognition that the very special conditions and remedies exhibited in the case of the Indian dams have a rather general application to the entire field of "pressure grouting"; yet the discussions, in the main, relate to experiences in sealing by cementation the foundation rock under dams.

The experience most nearly like those treated of in the paper appears in the discussion by Mr. Reed, wherein he describes the attempt to stabilize the Ringedalsvand Dam. It is interesting to note that although some benefits were accomplished by the cementation process, the results were not entirely satisfactory and the final remedy was found in the construction of an auxiliary curtain dam strutted against the water face of the leaky dam. This must have involved unwatering the reservoir, or the construction of expensive coffer-dams, either of which expedients was to be avoided if at all possible in the Indian situation.

The discussion by Mr. Fergusson is particularly apposite to the local problem of masonry construction with surkhi mortar. Doubtless, Mr. Fergusson has become aware of the growing use of Portland cement in substitution for hydraulic lime in his territory. It has recently come to the writer's attention that the cement manufacturing industry of India is expanding, with much activity in the promotion of sales. In 1931 the writer visited the just completed largest masonry dam in Hyderabad, Deccan, which was constructed of ashler-faced rubble in surkhi mortar, and was there informed by the Chief Engineer that he proposed no more important dams of that character, but intended to use Portland cement in the next one. In

NOTE.—The paper by D. W. Cole, M. Am. Soc. C. E., was published in February, 1935, *Proceedings*. Discussion on the paper has appeared in *Proceedings*, as follows: August, 1935, by Messrs. Oren Reed, F. F. Fergusson, and Joseph Wright; September, 1935, by Charles W. Comstock, M. Am. Soc. C. E.; October, 1935, by V. L. Minear, Assoc. M. Am. Soc. C. E.; and November, 1935, by James B. Hays, M. Am. Soc. C. E.

[10] Civ. Engr., New York, N. Y.

[10a] Received by the Secretary December 30, 1935.

this connection it may be suggested to Mr. Fergusson that useful possibilities may lie in the conversion of his kankar, or other local limestone, into natural cement rather than quick-lime, and then blending this cheaper form of cement with standard Portland cement for an economic mortar in dam construction. A lean mixture of this should prove better and but little more expensive than the surkhi.

Mr. Wright emphasizes the fundamental requirements for a safe and durable dam, a thesis which is affirmed by the experience described in the paper and which every generation of dam builders should take to heart.

The suggestion is pertinent that the "Christians" asphalt process might have been useful in staunching some of the worst runaway leaks to the lake as a preliminary to cementation. In retrospect, however, it would appear that the provision of all necessary equipment for this process in the distant and isolated locality would not have been justified for the limited requirement.

No dimension drawings are available, but the scale diagram in Fig. 1 gives all cross-sectional dimensions of Shirawta Dam as built. From this Mr. Wright will be able to analyze the stresses, using the tabulated weight of 146 lb per cu ft for masonry and such data for uplift, sliding factor, and cantilever integrity, as he will derive from the text of the paper.

The illuminating discussion by Mr. Comstock supplies a background of history, statistics, Oriental color, and technical wisdom against which the paper is most favorably displayed. Mr. Comstock's closing observation that: "A dam is made safe, or otherwise, on the job—not in the office", touches a responsive chord in the writer's constitution.

In the discussion by Mr. Minear it is gratifying to read that the Boulder Dam program of foundation grouting developed a system, independently, which is remarkably in accord with the routine in India, notwithstanding the dissimilarity of conditions and objectives. The check of one system against the other with harmonious results is indeed striking, tending to prove, for example, that both programs followed sound procedure in the situations as each was developed.

Mr. Hays raises a number of interesting points which it might be profitable to discuss if space permitted. Suffice it to say that the writer is in agreement as to the general superiority of pumps over the air gun for controlled grout delivery. Answering direct questions by Mr. Hays: The pistons of the air-driven pumps in the beginning were of cold-rolled shafting, 2 in. and 3 in. in diameter, with gland packing of a good grade of square braided hemp. Excessive wear of the piston (plunger) and the packing led to the substitution of nickel steel plungers, polished and packed in the same manner. These plunges were quite satisfactory and gave many weeks of wear, although much packing was consumed. Cylinder lining was unimportant since all wear was on the plungers with full bore packing. The distinctive feature of the pumps was the valve assembly consisting of ring-seated steel balls, as described under "Plan of Operation" in the paper.

The re-sifting of cement is an important consideration for work in tight ground, and in dams where limitation to low pressure is demanded. If again confronted with similar conditions the writer would plan to put the com-

mercial cement through a 200-mesh, rotating or vibrating sieve which could easily be rigged in the cement shed. This fine cement would be used in borings testing less than 10 gpm water loss. By this means the input of cement would be practically of colloidal texture and would probably accomplish about all that would result from application of the "chemical treatment".

The writer agrees that sand should be used sparingly, with caution and only in large cavities, or in runaway leaks which must be staunched as a preliminary to effective pressure grouting with thin mixtures. Sawdust, rice chaff, or other perishable materials were deemed inappropriate substitutes for sand in choking this type of leak in a structure where density was the prime essential. The writer agrees with Messrs. Hays and Minear that the best results can be secured by holding the pressure on the hole from start to finish, with the cement mixtures adjusted to give a maximum total input of solids before the hole chokes. It is not always feasible, however, and this is where experienced judgment, patience, and conscience have application.

In the writer's view a vast amount of "grouting" has been done in a perfunctory manner with ineffectual results, although reported as "holes grouted to refusal"; all for lack of intelligent and diligent supervision.

As to the multiple connection of holes for simultaneous grouting, it has been the writer's practice to require or permit this when pump delivery is demonstrably capable of holding the specified pressure on each individual hole of the series, usually not more than four holes at a time.

The discussion by Mr. Hays evidences appreciation of the requirements for effective sealing of difficult ground. There is a great variety of problems in the field, worthy of expert study.

An early example of foundation grouting is well recorded in much detail and with pertinent discussions in the paper by Harold A. Rands, M. Am. Soc. C. E., entitled: "Grouted Cut-Off for Estacada Dam".[11] A classic in tunnel grouting is the paper by James F. Sanborn and M. E. Zipser, Associate Members, Am. Soc. C. E., entitled "Grouting Operations, Catskill Water Supply".[12] Students of the general subject would do well to include these papers in their research.

[11] *Transactions*, Am. Soc. C. E., Vol. LXXVIII (1915), p. 447.
[12] *Loc. cit.*, Vol. LXXXIII (1919–20), p. 1058.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

UNDERGROUND CORROSION

Discussion

BY K. H. LOGAN, ESQ.

K. H. LOGAN,[19] ESQ. (by letter).[19a]—As one who has been in close touch with the soil-corrosion investigation since its beginning in 1922, Mr. Wolfe is quite right in cautioning against the unrestricted use of the data for the comparison of materials or the prediction of pipe life. In many of the soils under consideration the corrosion is not sufficiently serious to permit a reliable estimate of pipe life. In a few soils, however, all or nearly all the thinner walled specimens have been punctured. It is expected that the data on the 12-yr old specimens, which will be available in the spring of 1936, will show the standard deviation for each material. This will indicate whether or not there are significant differences between the rates of corrosion of the materials. As Mr. Baylis indicates, the causes and mechanism of underground corrosion need further investigation. To be of value any field investigation must involve a rather large number of specimens and, consequently, a very considerable expense for materials and labor. The outlook for the extension of the soil-corrosion investigation in the near future is not bright.

The relation between rate of penetration and time to which Professor Fetherstonhaugh calls attention, has been studied by several investigators. Dr. G. N. Scott[20] reached the conclusion that the relation for any soil could be represented by the equation, $P = \frac{UT}{B+T}$, in which P is the depth of the deepest pit at any time, T; B is a constant which is the same for all soils; and U is another constant different for each soil and equal to the ultimate pit depth.

Additional data recently obtained seem to indicate that some such an equation as that of Dr. Scott or Professor Fetherstonhaugh will represent

NOTE.—The paper by K. H. Logan, Esq., was presented at the meeting of the Sanitary Engineering Division, New York, N. Y., on January 18, 1934, and published in March, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1935, by Messrs. Thomas F. Wolfe, and John R. Baylis; and August, 1935, by Messrs. E. P. Fetherstonhaugh, John F. Skinner, and F. N. Speller.

[19] Chf., Underground Corrosion Section, National Bureau of Standards, U. S. Dept. of Commerce, Washington, D. C.

[19a] Received by the Secretary December 9, 1935.

[20] "A Preliminary Study of the Rate of Corrosion of Iron Pipe in Soils", by G. N. Scott, *Proceedings*, Am. Petroleum Inst., Vol. 14, Section 4, p. 212.

the progress of corrosion in well-drained soils, but that in poorly drained soils the depth of penetration is nearly proportional to the duration of the exposure.

The interesting experience of the City of Rochester cited by Mr. Skinner deserves further investigation. Although the superior service rendered by Conduit I may be due to the superiority of the material used, two other explanations are possible. The map showing the locations of the conduits which the writer obtained from the City Engineer's Office indicates that, although the two older conduits lie quite near each other, the older line lies at a slightly higher elevation and, consequently, is somewhat better drained. It is possible also that the protective coating applied to the older line was superior either as to material or application. This is suggested by the fact that at least some of the failures of the older line occurred at points where the coating had been injured. It is probable that the troubles on the newer line have been accentuated by its rather frequent exposure for repairs. A thorough investigation of the causes of the differences between the performances of the two conduits might be of great value to those who have to construct similar lines.

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DISCUSSIONS

FLOOD-STAGE RECORDS OF THE RIVER NILE

Discussion

BY JESSE W. SHUMAN, M. AM. SOC. C. E.

JESSE W. SHUMAN,[46] M. AM. SOC. C. E. (by letter).[46a]—The labor and patience involved in preparing a paper of this kind, are generally appreciated only by those who have made similar attempts, and the author deserves the thanks of the entire profession for his industry. As stated in the paper, the discharge records of rivers in the United States and in most other places extend back relatively only a few years. Although the Mississippi River, in length, is on a par with the Nile, the continuous discharge records run back only to about 1885; the records for the Tennessee River extend back about fifteen years more; so that if one wishes to study the behavior of a river, its mean, minimum, and flood flows, the available data upon which to base performance are really quite meager. In some cases the analyst finds that rainfall records over the drainage area of a river run back considerably farther than the discharge records, and through correlation between rainfall and run-off, he can "piece out" flows somewhat, and thus lengthen the record for statistical analysis. Although one might not believe in the theory of probability as applied to the study of river discharge, the longer the period embraced by the observations, the nearer to truth the results derived from them should be.

The Nile, alone, stands out as one river in the world on which flow data have been recorded for centuries. The author cites the various sources from which he has gathered the data depicted graphically in his paper. Doubtlessly, there are errors in the data—departures from the absolutely true data; but considering the nature of recording and bringing together these data in the first place, attended by the plus and negative errors of personal observation, it is quite likely that these, as well as the errors arising from copying and translation, would tend to neutralize one another, and thus leave an array of data that was fairly homogeneous, and suitable for any type of

NOTE.—The paper by C. S. Jarvis, M. Am. Soc. C. E., was published in August, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1935, by Halbert P. Gillette, M. Am. Soc. C. E.; and December, 1935, by Messrs. R. W. Davenport, H. E. Hurst, Thomas H. Means, J. W. Beardsley, and J. C. Stevens.

[46] Secy.-Treas., Power Eng. Co., Minneapolis, Minn.

[46a] Received by the Secretary November 8, 1935.

analysis. The Nile records, therefore, extend from the present back to about 620 A. D., with gaps here and there, but fairly continuously from 620 to 1521, and from 1700 to date, for the maximum stage of the river.

Except for a casual glancing over of the graphs of dots and various check marks, and drawing from them only general conclusions, no type of analysis can be applied to the data of the paper unless the values are "picked off" with a magnifying glass, which again introduces another error. The general conclusions that can be drawn are that the maximum annual stages were confined to elevations between 16 and 18.5 m for the entire period, 620 to 1500 A. D., with occasional wide departures; and that from about 1600, the stages were higher, culminating in high values about 1860 to 1880. The chart of minimum stages shows more fluctuation and depicts the secular rise in the river bed, as pointed out by Mr. Jarvis.

Using the magnifying glass, then, and a suitable scale (in this case, 40), one can determine the ordinate values of the charted data, and the information is ready for statistical treatment. The question arises, as to what can be accomplished.

There are many adherents of the practice of treating river-flow data as chance numbers, and the problem is left to them for their exposition. The writer is convinced that there is some law and order in river discharge (that the data are related numbers), and will endeavor to give his reasons therefor, followed by the type of analysis of river flow that results from such reasoning.

Consider the Nile River. It has a drainage area of about 1 107 227 sq miles. Although a record of the stages of the river is given over a long period, there is uncertainty, nevertheless, as to when any departures, increases, or decreases, may be expected or how long an apparent trend of values will continue. A search is made, therefore, for all available data that might possibly bear on the subject. Not much is really known about the Nile and its behavior, except that its source is now considered to be the Victoria Nyanza, with contributions from the Albert and Albert Edward Nyanzas, and the surrounding swamp countries close to the equator; that as it proceeds northward, it receives the discharge of the Sobat River, which drains a region of heavy rainfall; that farther north, the Blue Nile, rising in Abyssinia, pours its turbulent flows into the main stream, giving rise to the floods that have been recorded. The base, or steady flow, is provided by the White Nile, and the floods by the Blue Nile. Rainfall records are either lacking entirely or are scant, being for only a few years; so that little can be done by correlation with it as a factor. The River Atbara, rising in Northern Abyssinia, discharges into the Nile about 200 miles north of, and into the Blue Nile at, Khartoum.

An analysis of a river's behavior involves one of three methods: (1) Treating discharge data as purely fortuitous; (2) using harmonic analysis; or (3) using what is finally becoming known as cycle analysis. The disadvantages of the first method are that conclusions are drawn from data embraced only in the available record, for possible performance beyond that period; and that several eminent mathematicians have proved that the data are not chance numbers. The disadvantage of the second method is that, although one can break up into component elements a given curve of river flow, by means of the

Fourier series, the equations developed by the data will not necessarily reproduce the actual observed data at a later date. The disadvantages of the third or cycle analysis are, that it is as yet not completely developed, there being much to learn; that no treatise is yet available that covers the subject; and that, due to these causes, members of the profession have not become acquainted with it. It has some advantages over the other types of analyses, however, as the writer hopes to demonstrate in what follows.

The principles of cycle analysis will be developed as the writer proceeds with the treatment of the Nile data, but it is deemed necessary first to give some background to show that it is not extravagant nor lacking in consistency.

Briefly, cycle analysis consists of breaking up data into component elements, by suitable reducing means, quite differently than in harmonic analysis, and deriving certain pattern curves or cycles, which are similar to curves derived from sunspot data, giving thereby something to which terrestrial phenomena can be tied, and thus an inkling of what is to come and of what has gone before, beyond the periods of known data.

In order to make any progress, a research worker is obliged to set up certain postulates, for without his imagination, he would be merely an observer of phenomena and a gatherer of observational data. Therefore, he accepts the best that is offered at the moment. Clayton[47], Abbott[48], and others may be all wrong in their contention that the trigger action of the varying heat output of the sun is responsible for the changing weather. Clayton's work as head of the Weather Forecasting Department of the Argentine Republic, some years ago (where he successfully forecasted rainfall and temperature a week in advance) may not have been what it was purported to be, and the weather may be purely fortuitous or governed entirely by terrestrial causes; but even granting these points, it seems to the writer that there should be no objection to utilizing the fact that a certain cycle can be derived statistically from the sunspot numbers, varying in periodicity depending upon the length of two consecutive solar cycles, and which is named the Bruckner cycle[49]; and that also a similar cycle derived by similar processes in river run-off, rainfall, temperature, lake levels, etc., is identical in its general pattern, although varying in phase from place to place. This fact of phase variation, it seems to the writer disposes of much of the previously found confusing results.

It is found that the pattern of weather, as defined by the Bruckner cycle, is general, but with modifications as to intensity and fortuitous occurrence of yearly values, everywhere about the globe, and that the phase varies, depending upon the location, continentality, etc. For example, the Bruckner cycle detectable in the flow of the River Nile, is exactly opposite in its swings up and down, to the Bruckner cycle in the Mississippi River.

The question is frequently asked: "How can the changes in weather, which is different at various places, be brought about by any slight variation in the

[47] "World Weather", The Macmillan Co., N. Y., 1923.

[48] *Smithsonian Miscellaneous Collections*, Vol. 85, No. 1, and Vol. 87, No. 18.

[49] "Klimaschwankungen seit 1700", Vienna, 1890; also, *Monthly Weather Review*, Washington, D. C., July, 1926, and March, 1928.

solar constant, when the sun is more than 90 000 000 miles distant?" There are certain groups who maintain that it is not necessary to depart from this planet to find suitable agents that can cause weather to change. So far, however, this group has failed to bring forward any workable procedure for long-range forecasting, and it is certainly this very thing that the engineer is attempting when he analyzes the flow of a river which has records for only, say, fifty years; and yet he is trying to deduce its performance over a longer p riod of time in the future. Consequently, whether or not it is recognized, engineers are very much interested in long-range forecasting, in studying a river, designing a bridge, or laying out a great highway. The problem involved, is to ascertain and use, as key factors in design, not only the values in the past and present, but also what will be probable for some time to come in the future.

Studying the data of the Nile River as presented in this paper, with a view to utilizing them for cycle analysis, after several preliminary trials, it was decided to use the Lyons data[13] as charted from 1738 to 1872, and the Roda gauge data from 1873 to 1921, for the maximum annual stages, as well as the records taken from Fig. 3, giving the annual or seasonal volume of discharge at Aswan, from 1870 to 1932.

TABLE 6.—MAXIMUM ANNUAL STAGES AT RODA GAUGE

Date	0	Residuals		Scale and phase	Residuals		Scale and phase	$0-R_4=S_1$	M	R_4+M_1 = Double Wolf cycle
		R_1	R_2		R_3	R_4				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1 738	2.70									
1 739	2.00	4.70	.....	2.29						
1 740	2.45	4.45	915	2.21	4.50	.....	2.20	+0.25	+0.04	2.24
1 741	1.95	4.40	885	2.10	4.31	8.81	2.07	−0.12	+0.04	2.11
1 742	2.05	4.00	840	1.86	3.96	8.27	1.87	+0.18	−0.04	1.83
1 743	1.40	3.45	745	1.66	3.52	7.48	1.75	−0.35	−0.15	1.60
1 744	1.80	3.20	665	1.82	3.48	7.00	1.86	−0.06	−0.11	1.75
1 745	2.30	4.10	730	2.14	3.96	7.44	2.15	+0.15	0.00	2.15
1 746	2.15	4.45	855	2.50	4.64	8.60	2.30	−0.15	+0.10	2.40
1 747	2.40	4.55	900	2.06	4.56	9.20	2.03	+0.37	+0.10	2.13
* * *	* * *	* * *	* * *	* * *	* * *	* * *	* * *	* * *	* * *	* * *

In performing the computations to secure the cycles, the data are arranged, and manipulated as outlined, in Table 6. Values of the data are included only for 10 yr, in order to illustrate the method. Use paper ruled with vertical and horizontal lines, and enter in Column (1) the dates for the complete record to be analyzed. In Column (2) are the data from the chart after deducting 18.00 (so as to yield as small a number as possible with which to work); call these data 0. Now, add the first two values in Column (2), entering the sum in Column (3) opposite the second value as shown. This Column (3) is headed "Residual No. 1", or R_1. Now, repeat this operation, adding the two consecutive values of R_1, and writing the sum in Column (4), opposite the second value as before, these values being called R_2. R_4 can be derived by repeating the addition twice, and then dividing by 16, but after considerable work in this kind of manipulation, the

[13] "The Nile Flood in 1905", by Capt. H. G. Lyons.

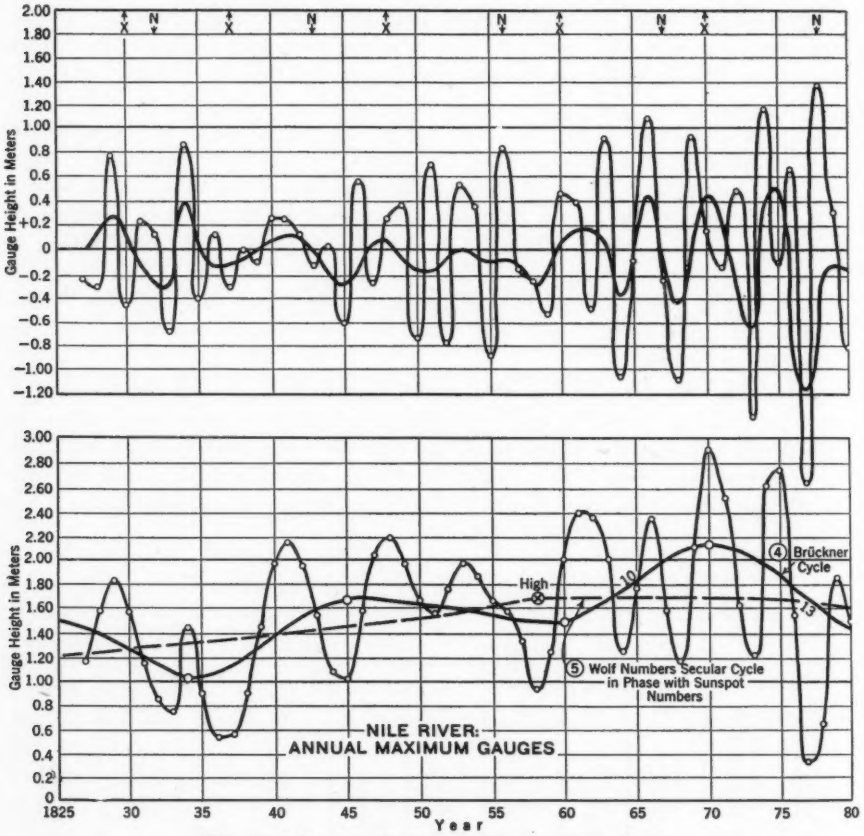

FIG. 10.—ANNUAL MAXIMUM GAUGES, NILE RIVER.

writer prefers to make only two additions, as shown, to R_2, and then to divide R_2-values by 4 and place the quotient one step upward in Column (5), headed "Scale and Phase". After R_4 is derived, it is restored to "Scale and Phase", and its values are entered in Column (8). Now, subtract the values of R_4 from the original value in Column (2), and enter the results, plus or minus, in Column (9), the values of which are called "Stratum No. 1", or S_1.

After the values of S_1 are secured for the length of the record, they are plotted as shown, for a representative part of the record, in Figs. 10 and 11. Although the original data began at 1738, the earliest value the residual yields is for 1740. Some care must be exercised in adopting proper vertical and horizontal scales, in order that the oscillating values of S_1 will show clearly, and also that the median line, M_1, drawn through the various loops of S_1, will be of sufficient scale that its ordinate values can be read with reasonable accuracy.

After S_1 has been plotted, as shown in Figs. 10 and 11 (Curve 1), one looks over the data of R_4 in Column (8), Table 6, and checks the maximum and

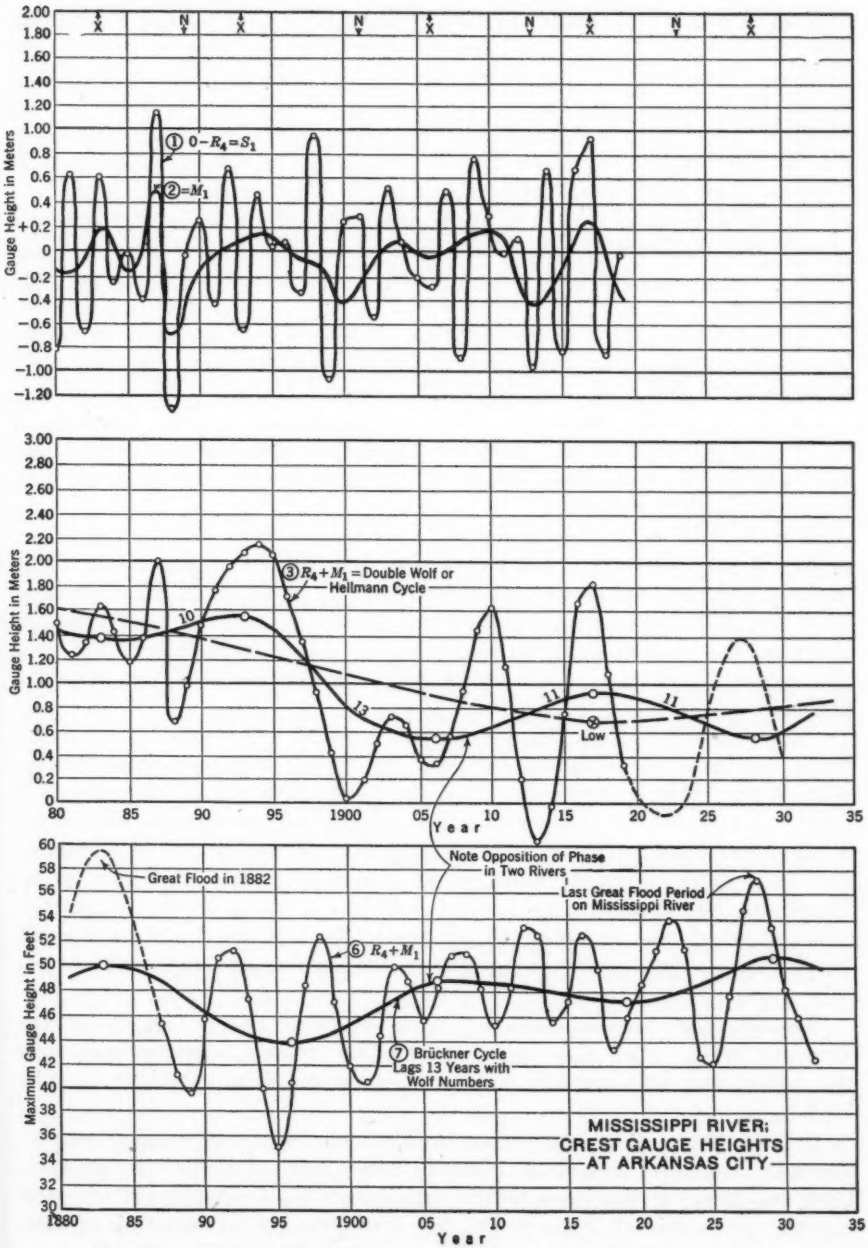

FIG. 11.—ANNUAL MAXIMUM GAUGES, NILE RIVER AND MISSISSIPPI RIVER.

minimum values which will alternate through the record. Next, the median line, M_1 is drawn through the curve, S_1, following as closely as possible, the "ups and downs" of R_4. The ordinate values of M_1, as now derived, are picked from the drawing, and entered as plus or negative values in Column (10), Table 6. Now, if M_1 is subtracted from S_1, the smallest cycle is secured, called the Clough cycle[50]. If M_1 is added to R_4 and the values are entered in Column (11), Table 6, the result is the Double Wolf, or Hellmann, cycle.

Before proceeding further it might be wise to offer a series of definitions, due to the undeveloped state of this type of analysis. The following cycles have all been discussed by various writers, but the notation has not yet been established.

Clough Cycle.—This is a small cycle of 2 to 3-yr periodicity, averaging about 28 months. It is very prevalent and persistent in all meteorological data, and was discovered by Clough, Clayton, and Abbot.

Wolf, or Wolfer Numbers.—These are provisional sunspot numbers; a Wolf number is an empirical index number, derived from the number and area of sunspots. The numbers are usually published after observational data have been slightly smoothed.

Double Wolf, or Hellmann, Cycle.—A cycle that crests about every 5.5 yr, generally just before, at, or after, sunspot maximum and minimum. It is variable and does not always appear consistently, although in some locations it is very clear, consistent, and definite. Hellmann was one of the first to point out the maximum rainfall that occurred twice in a Solar cycle. Horton also referred to it later. Some writers call it the Double Wolf, and some the Hellmann, cycle.

Solar Cycle.—This is the well-known cycle of the sunspot periodicity. From maximum to maximum sunspots, the period averages a little greater than 11 yr. This is commonly called the 11-yr cycle. It is detectable in a great many terrestrial phenomena and is especially prominent in lake levels.

Bruckner Cycle.—This is a cycle of twice the periodicity of two consecutive solar cycles, about 22 to 23 yr in length. It was first discovered by Bruckner[49], but by him given a varying length of 25 to 50 yr, the average value of which was set at ± 35 yr, due to the fact, as shown by Streiff[51] later, that his cycle was a composite of the secular and the cycle of two solar years.

Double Wolf Secular Cycle.—A cycle of twice the periodicity of the Bruckner cycle. First mentioned by Streiff in personal correspondence with the writer, and prominent in many meteorological data, especially in lake levels[52].

Secular Cycle.—The long swing in Wolf numbers is of varying length: The last high point was about 1856, the three previous highs were at 1798, 1704, and 1646. It is a cycle that is apparent in Wolf numbers, in tree-ring growth, varves, rainfall, temperature, lake levels, etc. Sometimes it is of such small amplitude as to be unrecognizable, in certain locations.

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[50] *Monthly Weather Review*, April, 1933.

[51] *Loc. cit.*, July, 1926, and March, 1928.

[52] "Notes on Lake Levels", by Jesse W. Shuman, *Monthly Weather Review*, March, 1931.

There are still longer and shorter cycles, but the foregoing are of the most use in analyzing the immediate past and future. Finally, looking back over the foregoing array of cycles, the observational data oscillate above and below the Double Wolf cycle, which, in turn, oscillates about the Solar cycle. This, in turn, oscillates about the Bruckner cycle, which, in turn, oscillates about the Double Wolf secular cycle; and, finally, the latter oscillates about the secular cycle of the Wolf numbers.

The values of $R_4 + M_1$, the Double Wolf cycle in the Nile River data, from Column (11), Table 6, are now plotted in Figs. 10 and 11, on the same time scale, as Curve 3. The break in the records from 1800 to 1825 is shown on the chart.

Now, after the $R_4 + M_1$ cycle is drawn, as in Figs. 10 and 11, it is sometimes necessary to reduce it further by two to four additional operations, before drawing in the median line, Curve 4, which is the Bruckner cycle. In this case, however, the further reduction seems to leave the relative vertical positions of the loops with respect to each other, unchanged. Therefore, the median line (Curce 4, Figs. 10 and 11) is drawn through the plotted data of $R_4 + M_1$. The median line, Curve 5 (the Secular cycle) is now dotted through the loops of the Bruckner cycle, the high point being at about 1860 and the low point, in Fig. 6, at about 1915 to 1920.

At the top of Figs. 10 and 11, under the scale of years, are placed arrows marked X or N, pointing upward or downward, to identify the times of sunspot maximum and minimum. The analysis is now complete and the analyst is free to examine what has been developed. It must be kept in mind that each plotted point of $R_4 + M_1$ (Curve 3), is a residual derived from five annual numbers, and that the actual yearly value may be more, less, or the same. In general, a peak of the Double Wolf cycle indicates that the average value of the five years' data—two years before and two years afterward—combined by residuation with the data for the plotted year, was at a maximum, and the troughs indicate the reverse.

Examining the Double Wolf cycle (Curve 3), it is noted that a succession of peaks and troughs, oscillate about the heavily drawn Bruckner cycle. At some places on the record they occur quite consistently with respect to sunspot maxima and minima, but by no means sufficiently so that one could rely upon the recurrence. The peak at 1861 occurs one year after sunspot maxima; at 1870, it occurs exactly on that date, as well at at 1883; but it is one year delayed at 1893, etc.; so that in the case of the Nile River, the Double Wolf cycle can be used for purposes of forecasting only in a general sense.

It is to be noticed, however, despite the secular rise in the river's bed, as depicted by the records of the minimum stages in Fig. 1 of the paper, that the period centering around 1870 represented a high point, or increased flood conditions, and that the period about 1905 represented a decreased condition of flow. The Bruckner cycle derived in the flow records of the River Nile, are just about, if not exactly, in phase with the Bruckner cycle in sunspot numbers.[53] The graph shows that the behavior of the Nile River is very

[53] For Bruckner cycle in sunspot numbers, see, the writer's Fig. 8 in discussion of "Rainfall Characteristics and Their Relation to Soils and Run-off", by C. S. Jarvis, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 95 (1931), p. 379.

strongly influenced by the Bruckner cycle. The data ended at 1921, and the last plotted date of $R_4 + M_1$ was at 1919. The question arises, can any of these cycles be extended into the future, with any dependability? The answer would have to be negative, if these data were all that were available. Fortunately, however, there are many other data on which to perform similar analyses, and results that can be compared with those of the Nile River. After a turning point has been developed in a cycle, it can be extended with considerable accuracy; hence the necessity of cross-identification with data from elsewhere, the phase of which may differ, and upon which a certain cycle may have reached a turning point.

The writer has taken data of tree-ring growth from Douglass[54], for a certain tabulation named "Flagstaff High (F.L.H.) 10 trees", and by using the same type of analysis, to derive the Double Wolf cycle, and, subsequently, drawing a median line through the various loops, the Secular cycle shown diagrammatically in Fig. 12 was secured. It will be noted that the wet

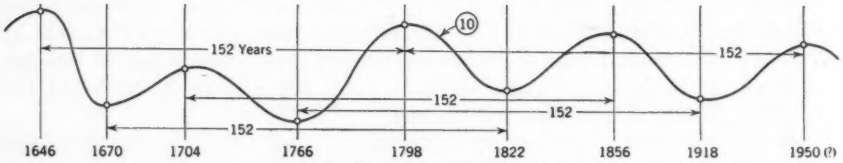


FIG. 12.—SECULAR CYCLE.

periods were at 1646, 1704, 1798, and 1856; that the dry periods were at 1670, 1766, and 1822, and that the last dry period centered around the date 1918. The tree-ring data did not extend far enough to get this last date, and until about 1932, the writer thought that this last date should be about 1905 to 1910; but examination of data from varied and widely placed sources, has led him to believe that the date must be about 1918. Then, too, another confirmation has only recently been given this date. A study[55] of Halbert P. Gillette, M. Am. Soc. C. E., on certain cycles, chiefly the 70 and 152-yr cycles, was pre-

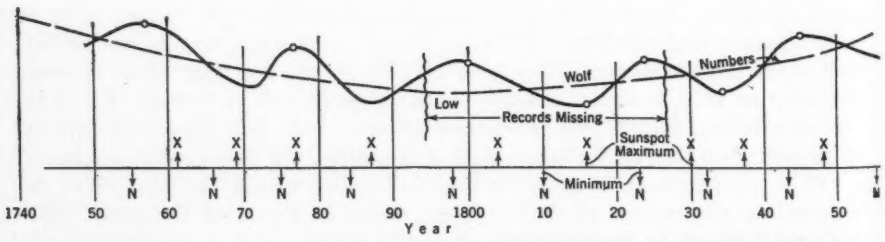


FIG. 13.—BRUCKNER AND SECULAR CYCLES IN

sented at Los Angeles, Calif., in June, 1935. The analysis as used herein will not develop this 152-yr cycle, but in examining Fig. 12, giving the Secular

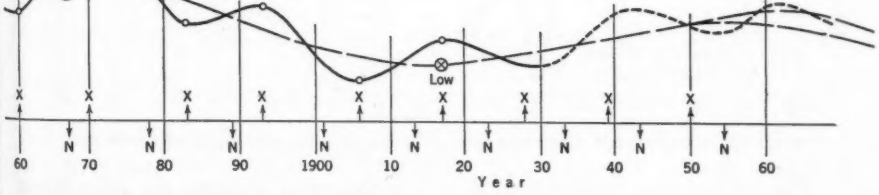
[54] "Climatic Cycles and Tree-Growth", by A. E. Douglass, Vol. II, The Carnegie Institution of Washington, Washington, 1928.

[55] "The Cycles That Cause the Present Drought", by Halbert P. Gillette, presented before the Annual Meeting of the Am. Meteorological Soc., Los Angeles, Calif., June 26, 1935.

cycle in tree rings, it will be noted that from 1646 to 1798 (two Secular cycles) is a lapse of 152 yr; from 1704 to 1856 (again two cycles) is 152 yr; furthermore, from 1670 to 1822 is 152 yr. This remarkable consistency, if genuine, would set the last low of the Secular cycle at 1766 plus 152 yr = 1918, and the next high at 1798 plus 152 yr = 1950. Every record of meteorological data examined recently by the writer seems to confirm this last low date as about 1918. This low is the turning point of the Secular cycle, which is threaded through the Bruckner cycles, so that there will be a low of the Bruckner cycle before, and after, the low of the Secular cycle.

Now, having some conception of what the Secular cycle is like, and the relative magnitude of its previous highs and lows, the student is in a position to draw some conclusions about the behavior of the Nile River, and to make some extensions beyond the times of the recorded data. Fig. 13 is a diagram, showing the Bruckner cycle of the Nile River from 1740 to 1960. The actual data analyzed ran from 1738 to 1921, with a break from 1800 to 1825. The diagram shows the highest period at 1870, the low period that must have occurred about 1800 (before and afterward), and the low period centering around the low turning point of 1918, that is, the low of the Bruckner cycle at 1905 and again the probable low of the Bruckner cycle at 1930. The next probable high period of maximum stages will center around 1940 to 1942 and conditions may equal those of the 1892–94 period; or, if the behavior is erratic, even those of the 1870 period. These ideas are indicated by the trends of the Bruckner cycle oscillating on the Secular cycle.

On Fig. 11, at the bottom, with the same time scale, is shown a graph of the Double Wolf cycle and the Bruckner cycle, in the Mississippi River data, taken at Arkansas City, Ark., for annual maximum gauges. The data on this river run back to only about 1885. Note that the Bruckner cycle in this case, is just opposite in phase to the Bruckner cycle in the Nile River. During the early Eighties, the Mississippi River had severe floods; whereas the Nile River was lower then than before or afterward. The last great flood of the



MAXIMUM ANNUAL GAUGES, NILE RIVER.

Mississippi River was in 1927, a period that must have had relatively moderate stages on the Nile River. In other words, the Bruckner cycle in the Nile River, is ahead of (leading) the Bruckner cycle in the Mississippi River by one-half the Bruckner cycle period—about 11 to 13 yr. The performance of the Mississippi River, in general, can be forecast from the behavior of the Nile.

For example, the high stages on the Nile in 1870 were followed in the Mississippi River about 1882 and 1883; the low stage of the Nile in 1883 was followed on the Mississippi River at about 1895, etc. Judging from past performance, the comparison of the Nile River and Mississippi River cycles, indicates that the high stages of the Nile in 1917 would be followed on the Mississippi River about 11 to 13 yr later, with real flood stages. The great flood of 1927 on the Mississippi River, came just ten years later. The low condition of the Nile in 1927 would lead one to expect a downward trend of the Mississippi River flood conditions until about 1940 (bearing in mind, however, that frequently a real flood occurs close to the low point of the Bruckner cycle, as at 1897, etc.)

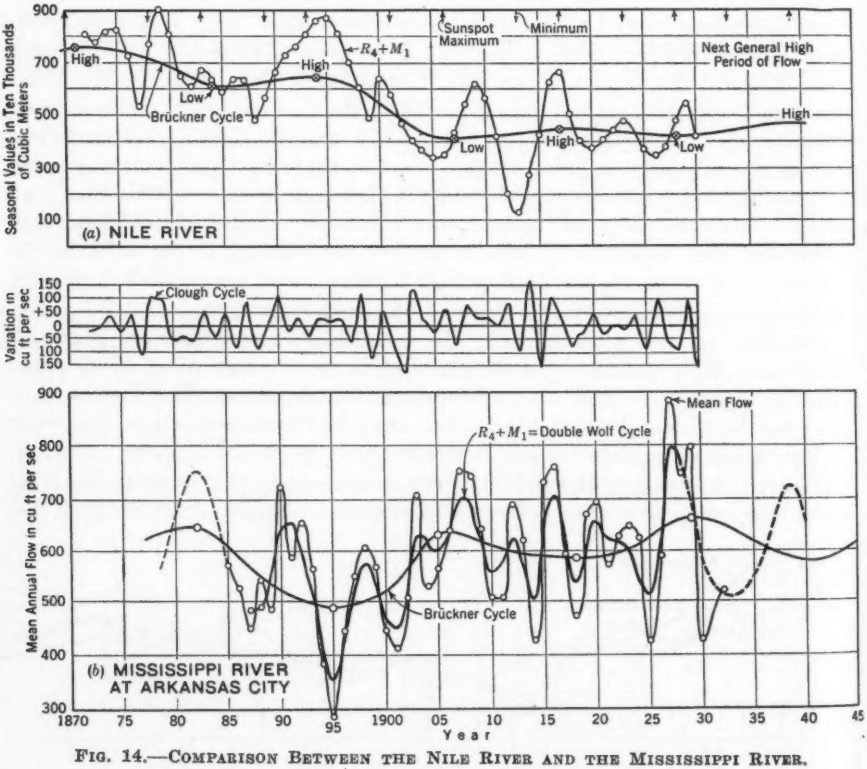


FIG. 14.—COMPARISON BETWEEN THE NILE RIVER AND THE MISSISSIPPI RIVER.

Fig. 14 has been prepared to show a comparison between the Nile River and the Mississippi River, using for the Nile, the data from Fig. 3, of seasonal discharge at Aswan, and for the Mississippi River, the mean annual flow at Arkansas City. In the Nile graphs (Fig. 12) the Bruckner cycle lags about one year with the same cycle in sunspot numbers, or is nearly if not in phase (although the writer has placed small circles at each turning point of the Bruckner cycles shown in all the graphs in this discussion, it must be under-

stood that it is probably impossible to define the actual turning point so accurately, at least with the type of mathematical procedure used herein. The circles are placed simply to give accent to the change of the trend in that region.)

Although the Secular cycle has not been drawn in, an inspection readily indicates that it is present in the data. For the Nile River, the Clough cycle is shown, secured by subtracting M_1 from S_1 (not shown herein). The algebraic sum of $R_4 + M_1$ and the Clough cycle equals the yearly discharge value.

The same cycles for the Mississippi River are shown in Fig. 14(*b*) as well as the annual values connected by a lightly drawn line. In this case, as in the case of Fig. 11, the same opposition in phase of the Bruckner cycles is in evidence. The Bruckner cycle is trending downward to a low point about 1940, although a peak in value of $R_4 + M_1$ may be expected before that time, of a magnitude considerably less than that in 1927.

Graphs[56] illustrating the varying level of Victoria Nyanza, the source of the White Nile, and the variation in the rainfall of that region are shown in Fig. 15. The sunspot maxima and minima are marked on the time scale,

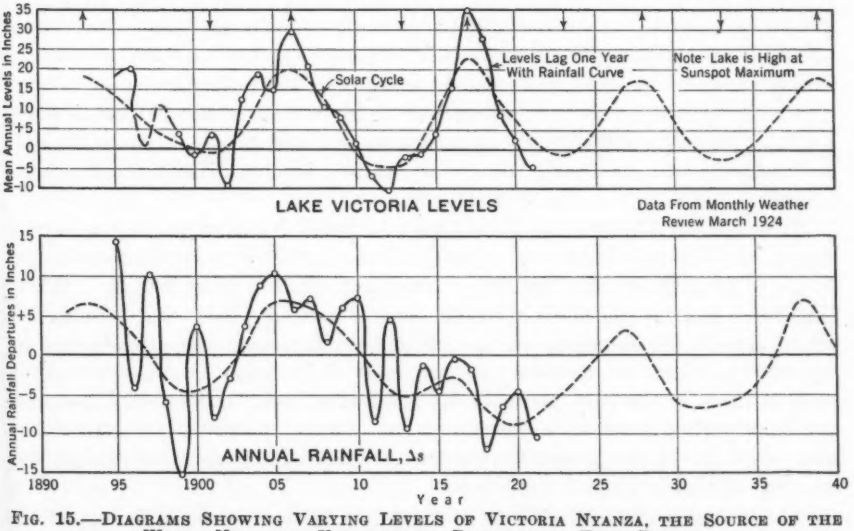


FIG. 15.—DIAGRAMS SHOWING VARYING LEVELS OF VICTORIA NYANZA, THE SOURCE OF THE WHITE NILE, AND VARIATION IN RAINFALL IN THAT REGION.

and the dotted line drawn freehand through the lake levels is the Solar cycle, which is beautifully clear. Note that the lake levels are at peak always at the sunspot maximum.

The outflow of this lake forms the base flow of the Nile, but there appears to be but little correlation, as would be expected, between the graphs of the river and the lake. If, however, comprehensive rainfall records were available, distributed over the drainage area of the Nile River, their statistical treatment taken in conjunction with that of the river, would yield immensely

[56] Original data pub. by Mr. C. E. P. Brooks, and later reviewed by Alfred J. Henry in *Monthly Weather Review*, U. S. Weather Bureau, Washington, D. C., March, 1924.

important information. This importance would be further enhanced if, in addition, records for temperature were included, and similarly treated, because temperature is a primary factor, whereas rainfall is secondary. For this lake Brooks found[57] that the correlation coefficient between the lake levels and rainfall was + 0.92, and between lake levels and sunspots, + 0.80.

The annual flood stages of the Nile River are of great importance to Egypt, and in bringing together the data described in his paper, the author has placed at the disposal of the profession valuable information, gleaned at great pains, which should ultimately lead to new interpretations. If mean monthly flow records were available for a period of years, this type of analysis would enable one to forecast the probable mean flow for the ensuing future year with considerable accuracy. In this discussion the author's records have been treated only back to 1738, and it is believed that sufficient knowledge can be deduced from this range of values because, although the regimen of the River Nile may be affected by the various causes, as pointed out by the author, the writer wishes to suggest that any such change in behavior would be gradual; so that in comparing values of to-day with those of 4 000 yr ago, when the average annual flood stages were about 23 ft higher than at present, the differences encountered would be of historical interest only. One would not need to consider the old records in designing a dam or other controlling works, which will endure probably not more than 150 to 200 yr.

Cycle analysis presents a conception of the variation in weather as registered by the river's flow, and suggests that extreme values, high or low, are not suddenly brought about, but are approached in an oscillating, pyramiding manner, culminating at the extreme value.

It is not yet 450 yr since Columbus discovered America, so that no structure in this country is that old. Prime movers are likely to become obsolescent in about 25 to 30 yr, and the life of a concrete dam is 50 to 200 yr. If any confidence can be placed in cycle analysis of a river's flow, that attempts to relate it with the changing weather, it follows that only values embraced in, say, 200 to 300 yr, are necessary for any type of analysis. Great catastrophies such as those in the past, probably did not strike from "a clear sky", but were the culminating point in a stormy period of greater intensity than normal, and which developed gradually enough so, that with proper knowledge, due warning of their approach was given.

The Ministry of Public Works of Egypt, in discussing the hydrology of the Nile Basin, states[58] that these records of the flow of the river have been analyzed for periodicities, and periods of small amplitude have been found. Quoting:

"They are, however, so masked in irregularities as to be useless for forecasting purposes. A connection exists between conditions in the South Atlantic and the Nile, but up to the present it has not been possible to make a forecast of the flood which is sufficiently reliable for practical purposes. However, as the science of meteorology develops and the nature of the mech-

[57] *Geographical Memoirs No. 20*, Air Ministry, Met. Office, London, England, 1923.

[58] "The Nile Basin", Vol. I, *Physical Dept. Paper No. 26*, Govt. Press., 1931, Cairo, Egypt.

anism producing the flood becomes known in detail, it may be possible some day to produce an accurate forecast of the nature of the flood some months in advance * * *."

The writer has already called attention to the connection that must exist in weather at different places, the reflection of which is given in river performance (for one index). It appears from the foregoing quotation that the authorities in charge of the controlling works of the Nile are cognizant of this fact and are working with it. The connection in the long swings, as defined by the Bruckner cycle, have been demonstrated in Fig. 11 between the Nile and the Mississippi Rivers. Doubtlessly, further search would bring to light other connections.

If the entire record of the Nile back to 620 A. D. were analyzed to secure the cycles described herein, interpretations of their probable effect upon the economic life of that region could be made by recourse to the vast storehouse of ancient literature that exists, which mentions here and there conditions that are ascribable to the Nile's behavior at the time. This river alone seems to have not only a very long record but also the written history of the times to accompany it, so that the field should be very fruitful.

Pharaoh Necho II (610 B. C.) was the first ruler[59] to attempt to create a connection between the two seas (Mediterranean Sea and Red Sea). Herodotus records this effort. The canal tapped the Nile at Bubastis, near Zagazig, and mention is made that the Red Sea extended farther inland then than at present. The length of this canal was computed by Pliny to be about 57 English miles. Herodotus states that its construction cost the lives of 120 000 men. Due to an adverse Oracle, warning that it might provide too easy means for approach to Egypt, the project was abandoned. Note that this was a period of ample water on the Nile. The canal was restored under Ptolemy Philadelphus (285 B. C.), and locks and sluices were used. This period also had ample water. Under Cleopatra (31 B. C.) the canal was impassible (a low-water period). Emperor Trajan (98–117 A. D.) found the Pelusiac Branch of the Nile very low, with practically no water. He tapped the Nile very much farther up stream.

Ptolemy wrote, in 150 A. D., the first clearly intelligent account of the origin of the Nile, something that had always been a great mystery. He writes of the two lakes, Victoria and Albert Nyanza, and the Mountains of the Moon. No less than 1 170 yr had to elapse before justice could be done this ancient geographer, and his account verified.

When one scans the history of regions not greatly remote from Egypt, and notes the vicissitudes of climate they have endured, it is significant that the records of the Nile disclose no such changes—merely changes of the weather. It was evidently with good reason that Egypt, with its droughts and famines notwithstanding, was a good place in which to live, by contrast with other not far distant regions, the flow of the Nile River being uniform enough so that even in its low periods, it carried sufficient water to keep the nation from migrating.

[59] "History of Egypt", by Rappoport, The Grolier Soc., London, England, 1904.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

THE STRESS FUNCTION AND PHOTO-ELASTICITY APPLIED TO DAMS

Discussion

BY MESSRS. ARSHAG G. SOLAKIAN, AND LARS R. JORGENSEN

ARSHAG G. SOLAKIAN,[26] ESQ. (by letter).[26a]—A description of the theoretical and experimental analysis of stresses in dams due to body forces and hydrostatic pressure is contained in this paper. Undoubtedly, it serves "to familiarize engineers with the use of the Airy stress function for the solution of problems in plane stress and plane strain * * *".

Much has been published in the United States concerning the optical principles involved in the art of stress analysis by means of polarized light and transparent materials, especially as applied to the determination of stresses in structural and machine elements. In this respect, the paper does not add considerably to the technique of photo-elastic measurements. Furthermore, the only experimental results offered are the boundary stress distribution curves and a photograph of an isoclinic band (which only serves to give the direction of the principal stresses along its path (Fig. 13)). The most useful patterns of stress fringes, which give a concrete idea as to the distribution of stresses in the model under specific types of loading, are omitted from the paper.

The method of conducting the experimental investigation (that is, of substituting body forces in the model by equivalent external loads) is not sufficiently accurate to justify its general adoption. It is for this reason that Mr. Brahtz finds that in certain cases (see following Equation (8)), in the upper two-thirds of the dam the stresses obtained agree with experimentation, and, in other cases, in the base region the boundary stresses are in close agreement with the approximate methods used in Part I. This indicates that for no special case of loading do the stresses in the entire field agree with those obtained by theory. An important objection to the superposition method of finding the resultant stresses (due to (1) the weight of the dam; (2) the vertical pressure of water over the top of the foundation; and (3) the lateral

NOTE.—The paper by John H. A. Brahtz, Esq., was published in September, 1935. *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1935, by I. K. Silverman, Jun. Am. Soc. C. E.; and December, 1935, by Fred L. Plummer, Assoc. M. Am. Soc. C. E.

[26] Lecturer in Civ. Eng., Columbia Univ., New York, N. Y.

[26a] Received by the Secretary November 26, 1935.

pressure of water against the face of the dam in contact), is that the principal stress orientations for the three different types of loading are different. Consequently, the investigator must go through a tedious analytical and graphical solution, which leads to results of questionable precision.

As Mr. Brahtz suggests, "the easiest method would be to rotate the model and observe it whenever it passes through the optical field". This is an exact method and has already been attempted with success in a similar problem, by making use of the centrifuge-polariscope apparatus developed at Columbia University[27], in New York City. A box containing the model of a retaining wall[28], with sand acting as back-fill, is allowed to rotate at constant speed to produce increased gravitational effects. A steady image of the stress fringes in the model is projected on the screen by means of a stroboscopic polarized light arrangement. The screen is stationary as in an ordinary polariscope of transmission type. The model and the sand are oriented in the box (see Fig. 15) in such a manner as to produce in the structure the resultant effect (as to magnitude and direction of principal stresses) of the weight of the dam, W_1, the vertical pressure of the sand, W_2, and the lateral thrust of sand, W_3. A fringe pattern[27] of the stresses obtained by this method, with a bakelite model, and sand back-fill rotating at a speed of 1 500 rpm in a centrifugal field having a gravitational strength of 485, has been published elsewhere[29]. Because of the limited speed and length of the rotating arm of the centrifugal machine and the size of the model made of bakelite having a moderate optical sensitivity to applied stress, the number of fringes produced in the pattern are not sufficient for an exact determination of the stresses. A new photo-elastic material[30], having an optical sensitivity about three times greater than that of bakelite, would have permitted sufficient data to be secured from a pattern obtained under similar conditions, for an exact solution of the problem.

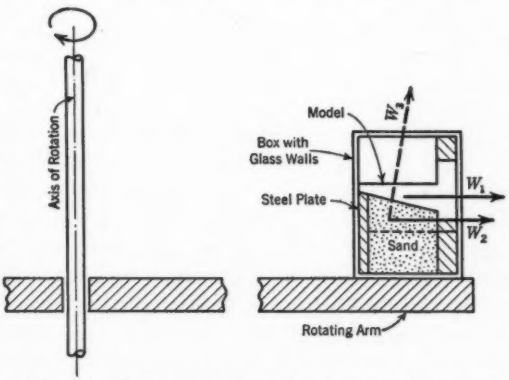


FIG. 15.—ARRANGEMENT OF MODEL AND SAND IN THE CENTRIFUGAL MACHINE.

LARS R. JORGENSEN,[31] M. AM. SOC. C. E. (by letter).[31a]—In adding to the knowledge of gravity dam design the author has done an immense amount of

[27] "Centrifugal Method of Testing Models", by P. B. Bucky, A. G. Solakian, and L. S. Baldin, *Civil Engineering*, May, 1935, p. 287.

[28] *Loc. cit.*, Fig. 3, p. 288.

[29] *Civil Engineering*, May, 1935, p. 290.

[30] "A New Photoelastic Material", by A. G. Solakian, *Mechanical Engineering*, December, 1935.

[31] Cons. Hydro-Elec. Engr., Berkeley, Calif.

[31a] Received by the Secretary December 16, 1935.

work. Except, perhaps, for the multiple-arch, and the Ambursen types, the design of dams is not yet an exact science, and there is actual need for more work in this field.

The actual safety of a gravity dam is made up principally of the following items. The weight acting downward on the foundation opposing any tendency to slide; the capacity of the material to withstand vertical tension at and near the up-stream face; and the shearing strength of the material, generally concrete. Since no gravity dam is known to have failed by crushing of the material, dangerous compressive stresses apparently do not develop in that type; this also checks with theory.

Since the vertical tensile strength of the bond between horizontal joints is a very uncertain quantity, just as is the shearing strength of such joints, although perhaps to a lesser extent, these items are ordinarily neglected, as they should be, when computing the safety factor of a gravity dam. For such a structure, the weight times the coefficient of friction must be greater than the water load, in order to give a factor of safety greater than one, when the structure is acting strictly as a gravity dam.

The writer has recently had an opportunity to make studies on a proposed gravity dam, 173 ft high, of triangular cross-section (top width, 20 ft) having a suggested base width of 75% of the height. In this case, the water load divided by the weight of the section was 0.55. Since the coefficient of friction may be between 0.6 and 0.75, this section acting as a gravity dam may have a factor of safety between 1.09 and 1.36, provided there is no uplift and no back-water. Such low factors of safety would not be considered in any other kind of a structure. On most well-built dams the factor of safety may be higher than the foregoing values indicate, but if it is, that fact will be due to the tensile and shearing strength of the concrete and not to gravity action.

Tests on models[32] have shown that stresses in dams are not linearly distributed as is generally assumed, but that tension does exist in the up-stream face of a loaded dam at and toward its junction with the foundation. These particular models had a base width of 77% of the height. The existence of tensile stresses in the up-stream face of a loaded gravity dam was also shown theoretically by Mr. B. F. Jakobsen[33], at least approximately, and by Ivan E. Houk, M. Am. Soc. C. E.[34]

A practical demonstration of the more than probable existence of tensile stresses in the up-stream face of a loaded so-called gravity dam is contained in the record of the failures of 68 gravity dams, with or without curvature, during the last 136 yr, or at the average rate of 1 every 2 yr. The best method for preventing such failures in the future is to accept the existence of these tensile stresses and provide gravity dams with a wider section than is customary at present.

The author shows a section of the Morris Dam on the San Gabriel River, in California. This dam would scarcely have any tension in the up-stream

[32] "Experimental Investigation of the Stresses in Masonry Dams", by J. S. Wilson and William Gore, *Minutes of Proceedings*, Inst. C. E., Vol. CLXXII, 1907–08.

[33] "Stresses in Gravity Dams by Principle of Least Work", *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 489.

[34] *Western Construction News*, April 10, 1932, p. 190.

face at and near the foundation, since its base is about 86% of its height. With its Levy cross-section, this dam is perhaps the most conservative gravity dam in California, and, undoubtedly, is safe under any condition, since the factor of safety of such a section always will remain greater than one, without having to depend upon any help from the tensile strength or shearing strength of the concrete.

How far one can go below a Levy section is a matter of judgment, but any reduction below such a section will be at the expense of the safety factor. Many engineers seem to have an unlimited faith in any dam called a gravity dam; undoubtedly, that confidence is well placed when the base width is 86% of the height, but for base widths of 66.6% of the height—which a considerable number of dams have—that confidence is badly placed. The actual factor of safety of such a section will depend upon the capacity of the concrete to take tension in the up-stream face. Gravity action alone will not produce a factor of safety worth mentioning.

Whether arch action increases the stability of a gravity dam arched in plan, and to what extent, at least in winter, is a debatable question and one not easy to answer. The question is even disputed internationally. The French regulations for building dams discourage the practice of curving gravity dams in plan, whereas the Italian regulations encourage it. The Italian regulations even go so far as not to require contraction joints, whenever a gravity dam is curved in plan.

The latest dam failure was on the Orba River, in Italy, where a gravity dam about 40 ft high went out on August 13, 1935, causing a great loss of life and property below it. A cloudburst brought down more water than the spillways could take care of and forced the surplus to flow over the crest. The section of the dam was quite ample for the normal water load, but not for the increased water load, which was estimated at 50% above the normal. This shows that the normal factor of safety could not have been more than 1.50 from all sources, since this factor of safety was completely cancelled by a 50% increase in the water load.

When a dam fails, there is generally much more involved than just the cost of the structure. It is especially important to satisfy the worst condition of stress distribution that can exist in the structure, rather than the best. Several of the recent gravity dam failures point to the fact that tension does exist in the up-stream face at, and near, the bottom of a loaded dam of less than the Levy section. The assumption of linear stress distribution, or of the author's stress distribution, is not on the safe side, although no fault can be found with the mathematics in the paper. If it was safe, several of the recent gravity dam failures would not have happened.

The earlier French regulations for building gravity dams contained the provision that, at all points of the up-stream face, such dam sections must have a pressure at least equal to the water pressure. This requirement produces the so-called Levy section having a base width of about 85% of the height. The new rules allow smaller sections, but for full reservoir there must still be compression in the concrete at the up-stream face.

Especially when it is subject to high-water pressure, concrete deteriorates with age, this deterioration being most marked in the up-stream section, where it is water-soaked. In many dams, deposits of lime have been shoveled out of the inspection galleries. This material can have come from no other place than the section of dam lying up stream of the gallery. The deterioration is most pernicious in its effect on the tensile strength, which is most needed relatively in this portion. If the section is so slim that the factor of safety depends in any great degree on the tensile strength of the concrete, then the dam is likely to be in distress at a fairly early age. The best means to avoid tensile stresses in the up-stream face is to provide an ample section. If such a section is much less than a Levy section, especially on the side-hill part of the dam, the factor of safety will only be an uncertain amount greater than one, acting as a gravity dam. Sixty-eight failures of gravity dams bear ample witness to this fact.

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DISCUSSIONS

THE RELATION OF ANALYSIS TO STRUCTURAL DESIGN

Discussion

BY MARSHALL G. FINDLEY, ASSOC. M. AM. SOC. C. E.

MARSHALL G. FINDLEY,[15] ASSOC. M. AM. SOC. C. E. (by letter).[15a]—In this paper Professor Cross has made a notable contribution to the literature of structural design. The problem discussed is not one of detail method, but of general balance, of plan of action, and of criteria for determining the choice of method. It might almost be said that the author presents a pragmatic conception of structural design. The problem involved needs discussion. So much literature of structural analysis has arisen that any one supervising design work is continuously faced with the need for decisions as to how far to go in the use of analytical methods. There is no question but that, on the one hand, certain practical problems are analyzed by mathematics too completely and too well. On the other hand, other problems are too little analyzed—solved too much by rule-of-thumb methods. Only rarely is analysis checked by common sense so as to keep it within bounds while making full use of its possibilities. The personal element complicates the practical problem, one designer enjoying the mathematics of theory too much, while another relies too much on experience, which may be imperfect and misleading.

The term, "hybrid", as used by Professor Cross, however, is very unfortunate. The writer refuses to use this term, and will use instead "monolithic" as being more appropriate. The criticism is that monolithic action is natural and fundamental in some structures, whereas the term, "hybrid", suggests that it is a combination of two types of action. Professor Cross has recognized that the basic nature of monolithic action is his method of moment distribution, wherein every computation starts by assuming perfect fixation at each end of each member. At the same time, the use of this inappropriate term, "hybrid", seems to be bound up with a fundamental misconception. In general, scarcely a structure has been built in which true and natural action is not in some sense "hybrid". It would be impossible to avoid the design of structures that do not somewhere, or in some manner, deserve this term of

NOTE.—The paper by Hardy Cross, M. Am. Soc. C. E., was published in October, 1935. *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1935, by Messrs. L. J. Mensch, and Russell C. Brinker.

[15] Structural Designer, Water Purification Plant, City Engr.'s Office, Milwaukee, Wis.

[15a] Received by the Secretary December 14, 1935.

opprobrium. The real point at issue is not whether "hybrid" structures can or cannot be avoided; it is rather how best to take account of actual monolithic action and its effects.

The criticism of this term is only incidental, of course; but in order to clarify the misunderstanding that it suggests, and also to lead the way to an examination of certain detail problems, the writer presents first some general considerations. These thoughts are partly a development of ideas expressed in Professor Cross' paper—one concerning the need of reconciling discordant views held by practical designers and by theoretical analysts; the other that the designer often tells a structure what to do, and then must see to it that it can do it efficiently. Professor Cross makes a division and then pays very little attention to deformation stresses, to participation stresses, or to so-called normal structural action. He stresses the increasing importance of monolithic action, and in this point all will agree. It is because of this increasing importance that structural analysis is becoming more widely used and, at the same time, more complex.

Two Conceptions of Structural Analysis.—Structural analysis is often considered an exact science. The aim under this conception is to secure an exact picture, by means of computation or of graphic analysis, of action of the beam or structure under stress; of its deflection at each point; of its tension, compression, or shear at this point; or of its dangers of cracking, twisting, or crushing, etc. This conception is very commendable within limits, but its limits are narrow. Only beams simply supported, simple continuous beams, deep trusses, simple frames, and a few other structures can be analyzed in this complete and thorough manner; and, usually, these structures will not be so analyzed for a practical reason—that is, the lack of time. Where such complete analysis is possible, imagination and experience usually make most of it unnecessary.

A larger conception of analysis must be sought, partly on account of current lack of knowledge, and partly on account of one's inordinate ambition to build complex things and to meet complex situations. Columns—simple and compound, flat-slab systems, complex frames, and various other structures that cannot be completely analyzed at present—must be designed. Design cannot wait for perfect analysis; as a matter of fact, it never has done so. Design has always led the way for analysis. Properly understood, design requires the larger conception of analysis. Analysis is no longer a complete and exhaustive prophecy of stresses and deformations. Instead, it must be relied upon as a basis of rational comparison whereby complex and unusual conditions may be assimilated to simple and ordinary conditions.

As a matter of fact, the actual progress of structural analysis is such that even approximate prophecy of action in many structures is impossible. Especially is this the case where pressures against, or produced by, earth are involved. No one knows exactly what happens in a complex footing or in a wharf bulkhead. Only on the larger conception of design is it reasonable to use mathematical analysis in connection with such structures. Analysis is then a guide to rational comparison between a proposed design and an existing structure somewhat similar. Such structures include columns, foot-

ings, flat-slab floor systems, steel and timber sheet-piling, and monolithic frames of concrete. The computation is a guide to design, but cannot be considered exact prophecy.

This larger conception of analysis, as the basis of comparative design, as the rational method of transfer of experience from shorter spans to longer spans, from narrow panels to wider panels, from light loads to heavy loads, from static forces to moving forces, etc., needs discussion and elaboration. The outmoded prejudice against indeterminate frameworks, still often expressed verbally, is clearly a consequence of the conception of analysis as exact prophecy. The current interest in frame analysis, in shrinkage and secondary stresses, and in soil mechanics, is a sign that the conception of analysis as a basis of rational comparison is in the ascendency. This interest sometimes leads to the assumption of impossible tasks, with resultant discouragement. Perhaps a discussion of certain definite problems, from the broader point of view, criticizing methods in the light of purposes, may be of some benefit.

The Simple Beam.—The outstanding case of "normal structural action" is perhaps the beam of one span simply supported at each end. In this case, when loads are given, shear and moment and deflection diagrams can be drawn quickly, and, superficially considered, prophecy of action might be considered quite definite. Theoretically, however, prophecy is not at all exact, and even practically there may be some question.

The fundamental assumption of beam theory, that a plane section before bending remains plane after bending, is only approximately true. In a rectangular metal section, it might be quite accurate, but in steel I-beams or channels, in timber with its grain and flaws, and in reinforced concrete with its two different materials, this assumption is only justified by the practical need for it. Shear computations are very rough. Even in a homogeneous rectangular section, it is necessary to use an approximation to estimate the relation of maximum unit shear to average unit shear. Furthermore, deflection computations usually neglect shear deformation entirely, whereas in some types of beams it is of greater magnitude than moment deformation; that is, even in a simple beam all the computations for shear, moment, and deflection are based on assumptions which are usually approximately true, but which, in exceptional cases, may be very far from the truth.

This fundamental assumption of beam theory is only supposed to hold true when no stresses exceed the elastic limit of the materials. This elastic limit is itself a more or less uncertain value, more variable in some materials than in others. Hooke's law of elasticity, that stress varies as strain, may become invalid in particular cases for various reasons, such as: (1) Overloading a member; (2) heterogeneity of a material (as in timber); or (3) a gradual departure from the straight-line ratio (as in concrete).

Certain practical conditions may also arise where the difficulty is more serious. Consider particularly the matter of lateral stiffness. Girders carrying rails across track hoppers, for instance, require more attention to lateral bracing than to routine moment and shear computations. From the designer's point of view the lateral forces acting are due to chance and cannot be esti-

mated except empirically. Cross-bracing or diaphragms introduced to stiffen such girders make them "hybrid" structures, which must act monolithically. In an ordinary floor the beams are braced at their top flanges by the slab. That this occurs is recognized in practical design by certain rule-of-thumb devices (cross-bracing, tie-rods, top cross-bars, etc.). The effect of these devices is to make a "hybrid" structure of even ordinary floors.

Thus, in general, even the simplest types of structure may exist under conditions that make them complex in action, and unsafe to design as by "normal structural action". Eccentric loads, the mechanics of buckling, and the relation between shear and tension, are not very well understood in fundamental theory, and must be dealt with by constant watchfulness on the part of the practical designer. "Hybrid" action which must be taken into account in some manner exists in many unsuspected details. With this warning in mind, moment, shear, and deflection diagrams serve as very satisfactory guides for the design of simple beams. However, they are only approximations. Materials do not conform thoroughly to the assumptions of design, even in the most favorable cases. Lack of homogeneity in material may result in eccentricity and, therefore, in so-called "hybrid" action, where the assumptions are practically perfect in every other element, and the designer must be on the lookout for such possibilities, even in the simplest cases.

Even a strut or tie designed for axial stress only, presents many uncertainties when examined inch by inch (who knows the exact details of internal stresses in a tension member around the rivet holes, for instance?); so that, in general, in plain tension or compression members, as well as in simple beams, there are uncertainties of fundamental assumption, uncertainties of homogeneity, uncertainties of rigidity, uncertainties of local stress, and uncertainties of perfect elasticity which make the theoretical part of the design very imperfect, considered as exact prophecy. In this case a generalization may be made. Even in the simplest cases, theoretical structural analysis may be considered successful only when, and because, it is a guide to safe, economical, and efficient design.

The Long Column.—The foregoing uncertainties apply to complex structures just as surely as to simple structures; but there are also other uncertainties of many kinds. One of these uncertainties appears in the case of the long column. Briefly, the best method of analysis seems to be an indirect one; namely, the selection of a section such that under chance small eccentricity, the tendency to bend does not increase with time. Other methods of analyzing long columns are not truly methods of analysis, but rather methods of empirical design.

In this case the rational purpose of design is that chance deformation stresses shall be controlled so that they do not essentially tend to increase with time. Here, clearly, if one small part of a column is weakened unduly, the result will be to throw enormous stresses into other parts of the column. Returning to the Cross definition of hybrid action, then, one may conclude that any long column is a "hybrid" structure, which, of course, is simply another way of stating that prophecy is difficult, as regards the exact action of a long column.

The Monolithic Frame.—The uncertainty of prophecy is complicated by an entirely different type of difficulty in the case of the monolithic frame. Assume for the present that all external forces are definite, known, and static (which is rarely the case, as, for instance, on a track culvert). The argument as to whether clear opening or center distance shall be used for each span will perhaps never be decided finally. Theoretically—that is from the point of view of prophecy—this is a very vital question. If one decision is made, end moments and shears will be computed, which are very large and which cannot possibly develop; and, consequently, the ends of the moment and shear diagrams must be adjusted or neglected. If the other decision is made, the question must be more or less arbitrarily settled as to the treatment of certain axial and eccentric forces acting on frame walls.

Practically, this question is not such a serious one, but only because the point of view of prophecy must necessarily be abandoned. Whichever decision as to method he makes, the designer follows that method through various structures, and learns what adjustments to make in order to obtain a satisfactory design. His analysis is of value as a basis of comparison, not as a prophecy. Enough structures of the kind have been designed, built, and tested to serve as a starting point for the comparison.

Deformation Stresses in Concrete.—Setting, shrinkage, and temperature changes normally affect all concrete structures; ordinarily, of course, the object is to distribute deformations, assigning them by definite joints to definite locations.

In certain cases, however, such changes cause participation stresses, thereby approaching "hybrid" action. Such a case arose in the design of certain structures for the Water Purification Plant, at Milwaukee, Wis. In this case, water-tightness requires continuity of certain walls and slabs, whereas the fact that certain slabs frame into a wall at different elevations means that temperature changes in the slabs will throw bending stresses of considerable magnitude into this wall. Only by a systematic comparison of some kind between temperature deformations in these slabs, and unit-moment deformations in the wall, can the resulting moment in the wall be determined. The stiffer the wall, the greater will be the resistance to temperature deformations, and the greater the moment in the wall and the tensile stress in the slab.

The principle involved, that of comparison of deformations, of course, is fundamentally the principle of all monolithic action. All methods of indeterminate structural analysis, whether graphic or algebraic, whether based on conjugate joints or the slope-deflection principle or the theory of least work, are methods of discovering distributions of stresses for which certain deformations are equal and consistent. If the writer understands correctly, Professor Cross looks upon simple and obvious uses of the principle of deformations as cases of "normal structural action", whereas more complex uses of this principle are covered by the term, "hybrid action".

To this point, elements of uncertainty that must be taken into account by structural analysis, even including the so-called "hybrid action", have been discussed; but all the cases examined thus far have assumed known and

definite loads acting on the structures in question. Other elements of uncertainty exist, which the writer wishes to include in this discussion, in which external loads are increased or decreased by the action of the structure analyzed. Every designer has analyzed some of these cases, sometimes by guess, sometimes by rough assumption, sometimes by more careful analysis which, however, is always open to question as to its meaning. The writer believes that analysis of these cases is usually practically possible, and, in general, should be undertaken; but such analysis is clearly wasted time and effort if considered as exact prophecy.

Simple Footings.—The assumption commonly made is that earth pressure is uniform over the entire area of the footing (for centered loads) or that it varies uniformly from one corner or side to the opposite one. In the design of footings this assumption is nearly always on the safe side, so it is used very often although known to be erroneous. The more accurate assumption, that intensity of earth pressure varies as the deformation of its surface plane, is difficult to use in many cases, starting as it does with a differential equation of the fourth order. Practically, extreme theoretical accuracy is not required. Consider, for instance, a simple wall footing, with a centered load. The theoretical curve of earth pressures is a sine curve, but a parabola or an ellipse may be used without serious error; that is, the resulting shear and moment curves will be the same for practical purposes. In fact, a straight-line diagram may be drawn, which is also sufficiently close as an approximation.

In more complex cases, with two or more columns on a footing, or the introduction of wind or impact forces, an exact earth pressure diagram becomes practically, although not theoretically, impossible to draw. In some cases, practical approximations for design purposes can be made quite satisfactorily; in other cases apparently no amount of study and discussion will obtain even approximate earth-pressure diagrams, and the designer must analyze the structure by the method of limits, assuming several possible worst cases.

The situation is further complicated, of course, by the difference in action between sand, clay, hardpan, and rock, and by the usual lack of homogeneity in the foundation material. In this case, the essential point is that stress in the footing and stress in the foundation material are related to each other by means of deformation. The stress in each is affected by the capacity of the other to take stress, and not in small degree, but vitally. Thus, the load distribution cannot be known until the design is settled. This is really another type of structural action not included in the paper, and should be given a separate name in generalized discussion. Perhaps the term "reactive" action might be coined for this type of structural action.

The footing is by no means the most complex form of "reactive" action. There are at least two definite forms in which design analyses are being carried on where such "reactive" action exists in more complex form.

Sheet-Piling.—Especially as used in anchored cantilever bulkheads, for quays, and slips, the sheet-pile is loaded by active horizontal forces due to water and earth pressures, and is held in place by the passive resistance of the earth near the bottom, and by an anchorage at the top. The anchorage at the top, in turn, often is held in equilibrium by developing passive earth

pressures. Separating the items of loading and of structural action, then, the following items remain to be considered: (1) Water pressure, which is perfectly definite, and can be computed without appreciable error; (2) active earth pressure, which is definite but more difficult to determine exactly, and nevertheless can usually be approximated sufficiently well for practical purposes (like water pressure, this pressure exists independently of the deformation of the movement of piles); (3) passive earth pressure, usually greatest near the bottom of the sheet-pile (in some cases passive earth pressure can be estimated very closely by the conditions of static equilibrium, and, in other cases, it depends on Item (4)); (4) bending action in the sheet-pile, which must be considered because only by deformation or movement can passive pressure be developed; (5) deformation at the top of the pile, due to give of anchorage, axial stress in anchor rods, and bending of wales; and (6) passive pressure developed by the anchorage, which mutually depends upon and affects all these other items.

Theoretical methods have been proposed, which take certain of these items into account. Empirical methods have been used, which take all of them into account, although often by the cut-and-try method. It is clearly a difficult type of analysis, because pressures that can themselves be estimated primarily only by methods of indeterminate structural analysis determine, and in turn are modified by, pressures in which a secondary analysis by means of deformations must be made.

The term, "complex reactive" action, may be applied to this type of action. Design analysis of this type might be considered an attempt at prophecy of action, but as such is certainly not exact.

Roadway Slabs.—An element of contingency enters into the design of tall buildings to allow for lateral resistance in regions subject to hurricanes and earthquakes, or in the selection of sizes for culvert drains in regions subject to heavy rainstorms. The heaviest conceivable winds or earthquakes cannot be resisted economically; and the heaviest rains must be expected to cause floods. First, the designer must decide where to draw the line between high cost and awkward structure on one hand, and greater convenience and economy with less safety on the other.

A similar element of contingency occurs in the design of roadway slabs, in addition to the consideration of moving loads, impact, and "reactive" action. If all sub-grades were compacted perfectly and remained so, there would be little bending stress in any roadway slab. Modern construction methods attempt to produce a perfect sub-grade which will remain perfect. Consequently, although the designer may feel very sure that there will be failures of sub-grade, as a matter of experience, he cannot prophesy where a failure will occur, or how far it will extend. The assumption usually made is that a circular failure of a certain diameter will occur in the sub-grade, and that the slab should be reinforced to carry expected concentrations across such holes, whether central or at the edge of a slab.

In the selection of the diameter of assumed sub-grade failure, the designer is again drawing the line of contingency. In this case design analysis must cope with a third type of "reactive" action which might be termed, "contingent

reactive" action. Such design analysis is not prophecy even approximately, but only a basis of comparative design which may be standardized after testing by use.

Criticisms.—After consideration of these rather ordinary cases, it is obvious that Professor Cross' very excellent introduction to the subject needs some modification. He emphasizes two points about which there can be no question, and which are very enlightening and suggestive. The first is the generalization as to participation stresses, which, if not new, is certainly worded effectively, so that deductive reasoning can draw out again from it the fruits of experience which have gone into its formation. The second is the pragmatic conception of design where so-called "hybrid" action exists. Of course, the personal element enters in; nevertheless, it often occurs that the designer "tells a structure what he wants it to do, and the structure will try to do it."

It is because of the characteristic of monolithic and complex structures herein indicated, that safety in structural design can be obtained so often without lengthy theoretical analysis, and with only slight loss in economy. By the method of limiting cases, which is an attempt to prophesy several of the worst conditions that might occur at opposite extremes, it is often possible to design, safely and quickly, a very complex monolithic structure.

The writer has previously criticized the author's use of the term, "hybrid". Now, it seems in place to comment upon another use of words, namely, "normal structural action". This phrase is not so much of a unity as it seems because the author includes in it not only statically determinate structures, but also monolithic structures of varying degrees of complexity, similar only in that the experienced designer can approximate their true action without much study. In other words, design analysis as a method of prophesying actual action is applicable.

Conclusion.—In summary the writer wishes to emphasize several points which have been raised. The first is that "normal structural action" is not a convenient term; even the simplest static action includes many elements of uncertainty. Furthermore, it should be subdivided into static action, and simple monolithic action, which are quite different in their essential nature. "Hybrid" action is perhaps an excellent term for complex indeterminate action when a definite load is divided according to relative stiffnesses of two or more independent or partly independent structures or systems of members.

Certain new classes have been proposed which should be included, resulting in a revision of the classification by Professor Cross, as follows:

I.—Stresses Determined by Strains:

- (*A*) Deformation stresses; and,
- (*B*) Participation stresses.

II.—Types of Action for Supporting Known Superimposed Loads:

- (*C*) Static action;
- (*D*) Simple monolithic action;
- (*E*) Complex frame action;
- (*F*) Hybrid or interactive action; and,
- (*G*) Empirically analyzed action (columns, flat slabs).

III.—Type of Action in Which Loading Is Affected by Action ("Reactive"):

(*H*) Simple reactive action;
(*J*) Complex reactive action; and,
(*K*) Contingent reactive action.

The writer agrees with the author very heartily in his suggestion that further distinctions are needed in the general discussion as to the relative advantages of determinate and indeterminate systems; and hopes that Professor Cross will concur, at least in a general way, with the further distinctions suggested herein.

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DISCUSSIONS

TRUSS DEFLECTIONS: THE PANEL DEFLECTION METHOD

Discussion

BY MESSRS. DAVID B. HALL, AND E. MIRABELLI

DAVID B. HALL,[2] Assoc. M. AM. SOC. C. E. (by letter).[2a]—By well-chosen formulas, Mr. Shoemaker has unquestionably eliminated most of the superfluous operations and reduced the labor of calculating truss deflections to a minimum. However, these same formulas have obscured the underlying principles to a certain extent, so that the actual application of the method must consist mainly of performing the indicated operations faithfully and blindly.

The following solution of the arch truss (Fig. 5), although probably no shorter, may be, therefore, of some assistance in interpreting the paper, since each operation can be explained separately, and in fairly simple terms.[3]

In general, if a section cutting three members is passed through a truss, a change in length of one of these members causes the free side of the truss to rotate about the point of intersection of the other two members, and the amount of the rotation equals the change in length of the member divided by the perpendicular distance from the center of rotation to the member. Shortening Member 4–6 in this manner causes the left portion of the truss to pivot about Point 5, and the stretch in Member 4–5 causes it to rotate about the point where Member 4–6 (prolonged) intersects the top chord.

For chord members, since the center of rotation is at a panel point, the work is very simple and obvious. Thus, when Member 5–7 (Fig. 6) stretches 2.14 units, the deflection one panel length from Point 6 (the hinge for this member) is $2.14 \times \frac{25.0}{12.5} = 4.28$ units, and at more remote panels is a multiple of this value. In the case of lower chords, the lever arms of which are not given directly, it will be noted that the lever arm is equal to the

NOTE.—The paper by Louis H. Shoemaker, M. Am. Soc. C. E., was published in November, 1935, *Proceedings*. This discussion is printed in *Proceedings*, in order that the views expressed may be brought before all the members for further discussion of the paper.

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[2a] Received by the Secretary November 26, 1935.

[3] For a similar method see letter by William Bertwell, Assoc. M. Am. Soc. C. E., *Engineering News-Record*, November 1, 1934, p. 565.

height of the panel times the cosine of the chord inclination, and that the cosine of the chord inclination equals the panel length divided by the chord length. From this it readily follows that substitution of the chord length for the panel length is the only additional step necessary. Thus, the deflection factor for Member 2–4, for instance, is $\frac{27.9}{32.5}$, as given in Table 4.

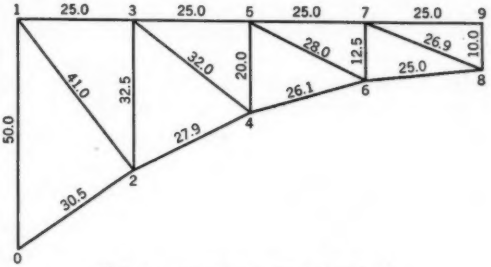

FIG. 6.—LENGTHS OF MEMBERS.

To avoid the inconvenience resulting from the imaginary centers of rotation for web members the following procedure appears to be the simplest to explain. The deflection at the end of a vertical web member is equal to the change in length of the member. For a diagonal, the change in length must be multiplied by the length of the diagonal divided by the height of the panel on the stationary side.

TABLE 4.—VERTICAL DEFLECTIONS

Member	Change of length	Factors		Deflections						Horizontal deflection factors	Horizontal deflection
		Base	Increment	Base	Increment	Panel Points					
						0 and 1	2 and 3	4 and 5	6 and 7		
7-9	2.33	.....	25.0÷10.0	....	5.83	23.32	17.49	11.66	5.83	4.00	9.32
5-7	2.14	.....	25.0÷12.5	....	4.28	12.84	8.56	4.28	....	3.00	6.42
3-5	1.39	.....	25.0÷20.0	....	1.74	3.48	1.74	.....	....	1.50	2.09
1-3	0.50	.....	25.0÷32.5	....	0.39	0.39	.....	.....	....	0.538	0.27
6-8	2.51	.....	25.0÷12.5	....	5.02	15.06	10.04	5.02	....	4.00	10.04
4-6	1.51	.....	26.1÷20.0	....	1.97	3.94	1.97	.....	....	2.61	3.94
2-4	0.857	.....	27.9÷32.5	....	0.74	0.74	.....	.....	....	1.72	1.47
0-2	0.555	.....	.....	....	....	.....	.....	.....	....	1.22	0.68
7-8	1.38	26.9÷10.0	2.5÷12.5	3.71	0.74	5.93	5.19	4.45	3.71	1.08	1.49
5-6	2.61	28.0÷12.5	7.5÷20.0	5.85	2.19	10.23	8.04	5.85	....	1.68	4.39
3-4	2.62	32.0÷20.0	12.5÷32.5	4.19	1.61	5.80	4.19	.....	....	1.23	3.22
1-2	3.02	41.0÷32.5	.....	3.80	....	3.80	.....	.....	....	0.88	2.66
6-7	0.42	1.0	2.5÷12.5	0.42	0.08	0.66	0.58	0.50	0.42*	0.40	0.17
4-5	1.25	1.0	7.5÷20.0	1.25	0.47	2.19	1.72	1.25*	....	0.75	0.94
2-3	1.66	1.0	12.5÷32.5	1.66	0.64	2.30	1.66*	.....	....	0.77	1.28
0-1	1.75	1.0	.....	1.75	....	1.75*	.....	.....	....	0.70	1.22
Upper panel points	.....	.....	.....	.....	....	90.68	59.52	31.76	9.54	....	49.60
Lower panel points	.....	.....	.....	....	....	92.43	61.18	33.01	9.96	....	.....

* Lower panel points.

The following proof of this rule may be found easier to remember than the rule itself. In Fig. 7, if Member BC (or a) changes in length an amount, Δa, while Side b remains fixed, Side c will rotate about Point A. The rotation will be $\frac{\Delta a}{h_a}$, and the vertical movement at Point B will be $\Delta a \frac{h_b}{h_a}$. Now,

$\frac{h_b}{h_a} = \frac{a}{b}$, since $b\frac{h_b}{2} = a\frac{h_a}{2}$ = area of triangle (also because $\frac{h_b}{a} = \frac{h_a}{b} = \sin C$). Therefore, the deflection at Point $B = \Delta\, a\, \frac{a}{b}$. Thus, the deflection at Point 5 for Member 5–6 is $2.61 \times \frac{28.0}{12.5} = 5.85$. Since the deflections at other

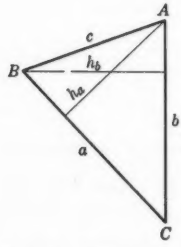

FIG. 7.—HORIZONTAL DEFLECTIONS.

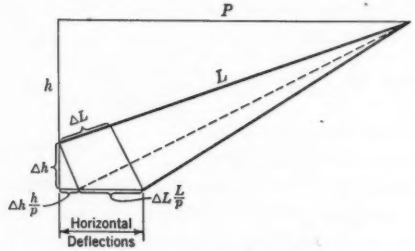

FIG. 8.—WEB MEMBERS.

points are proportional to their distances from the center of rotation, and since, in this case, the distances of Points 5 and 7 are in the same ratio as the panel heights at Points 5 and 7, it is seen that the difference per panel may be obtained by multiplying the deflection at Point 5 by $\frac{(20 - 12.5)}{20}$, making the panel increment equal to $2.61 \times \frac{28.0}{12.5} \times \frac{7.5}{20} = 2.19$. Thus, the deflection at Points 2 and 3 contributed by Member 5–6 is $5.85 + 2.19 = 8.04$, etc.

For a single horizontal deflection at the support as in this problem, a very simple method is available. The multiplying factor for each member of the truss is the stress due to a unit load at the support. These factors are already known, and the calculations are shown in the last two columns of Table 4. The remainder of the work, transposing the deflections and calculating the horizontal reactions, is the same as in the original paper and has not been repeated. For algebraic signs it is sufficient to examine the picture and note whether a given deformation deflects the truss up or down.

TABLE 5.—HORIZONTAL DEFLECTIONS, BOTTOM CHORD PANEL POINTS

Point	Δh-CONTRIBUTION				ΔL-CONTRIBUTION			Combined increment	Horizontal deflection
	Vertical deflection	Difference, Δh	Factor	Increment	ΔL	Factor	Increment		
8	0								0
		9.96	2.5÷25	1.00	2.51	1	2.51	3.51	
6	9.96								3.51
		23.05	7.5÷25	6.91	1.51	26.1÷25	1.58	8.49	
4	33.01								12.00
		28.17	12.5÷25	14.09	0.857	27.9÷25	0.95	15.04	
2	61.18								27.04
		31.25	17.5÷25	21.87	0.555	30.5÷25	0.68	22.55	
0	92.43								49.59

If complete horizontal deflections are required (as in the case of secondary stress calculations), the method illustrated in Table 5 may be used. Each bottom chord member has its ends displaced vertically a known amount and its length changed a known amount. The resulting horizontal movement is shown in Fig. 8.

E. MIRABELLI,[4] M. AM. SOC. C. E. (by letter).[4a]—The numerical examples chosen by the author to illustrate the application of the panel deflection method involve symmetrical loadings and symmetrical structures with vertical members at mid-span. In dealing with such cases it is possible to use the center vertical as a reference bar and thus to obtain deflections directly. When there is lack of symmetry it is necessary to apply corrections for rotation of the structure for the same reason that corrections are necessary when a Williot diagram is used. The necessity for such corrections when dealing with vertical deflections may be avoided by combining the panel deflection method with a method of elastic weights.

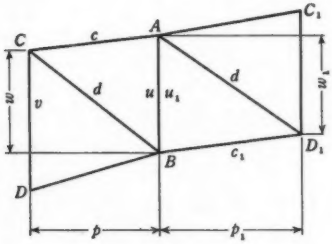

FIG. 9.

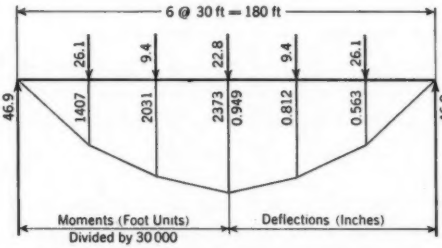

FIG. 10.—DEFLECTIONS OF A 180-FOOT SIMPLE TRUSS.

Referring to Fig. 9 which shows two adjacent panels of a truss, the elastic weight to be applied at Panel Point B may be expressed by the equation developed by Müller-Breslau[5]. In the notation used by the author, the elastic weight is:

$$W_B = -\frac{\Delta_D}{p} - \frac{\Delta_{D1}}{p_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

in which, Δ_D = downward deflection of Point D with respect to Point B; and, Δ_{D1} = downward deflection of Point D_1 with respect to Point B.

From the author's development (under the heading "General Formulas for Panel Distortions"):

$$\Delta_D = \frac{c'\,c - d'\,d + w\,u'}{u} + v' - u' \quad \ldots\ldots\ldots\ldots\ldots (9)$$

and,

$$\Delta_{D1} = -\frac{c'_1\,c_1 - d'_1\,d_1 + w_1\,u'_1}{u_1} \quad \ldots\ldots\ldots\ldots\ldots (10)$$

[4] Asst. Prof. of Structural Design, Dept. of Civ. and San. Eng., Mass. Inst. Tech., Cambridge, Mass.

[4a] Received by the Secretary December 9, 1935.

[5] "Die graphische Statik der Baukonstruktionen", by H. Müller-Breslau, Fifth Edition. Vol. 2, Pt. 1, Stuttgart, 1922, p. 95. Equation (2).

Equations (9) and (10) may be combined with Equation (8) to determine the elastic weight for any panel point of the truss. The deflections of the bottom chord panel points are then found by the usual method of calculating the bending moments in an auxiliary end-supported beam on which the elastic weights are applied as loads. It should be noted that Equation (9) is used for panels in which the diagonal is attached to the lower panel point, B, and Equation (10) is used for panels in which the diagonal is attached to the upper panel point, A.

The data from the author's Example 1 are used to illustrate the application of this method. The elastic weights are computed as shown in Table 6. The application of elastic weights to the auxiliary beam and the determination of bottom chord deflections are shown in Fig. 10.

TABLE 6.—ELASTIC WEIGHTS FOR A 180-FOOT SIMPLE TRUSS

Panel	Member	Length		Change in length	Products, $c'c$, $d'd$, wu'	Δ_D or Δ_{D1}	30,000 W_B
		Symbol	In feet				
2–0	0–2	c_1	30	+184.5	+5 530		
	0–1	d_1	43.1	−286.2	−12 340		
	1–2	u_1 (w_1)	31	+159.3	+4 940	−736	
2–4	2–4	c_1	30	+184.5	+5 530		
	1–4	d_1	43.1	+210.2	+9 060		
	1–2	u_1 (w_1)	31	+159.3	+4 940	−46	+26.1
4–2	1–3	c	30.4	−222.5	−6 770		
	1–4	d	43.1	+210.2	+9 060		
	3–4	u	36	+ 14.4	+ 446		
	1–2	v (w)	31	+159.3	.....	−282	
4–6	4–6	c_1	30	+189.3	+5 680		
	3–6	d_1	46.8	+132.8	+6 220		
	3–4	u_1 (w_1)	36	+ 14.4	+ 518	+1	+ 9.4
6–4	3–5	c	30	−220.8	−6 620		
	3–6	d	46.8	+132.8	+6 220		
	5–6	u (w)	36	0	0		
	3–4	v	36	+ 14.4	.....	−342	+22.8

It appears to the writer that the panel deflection method may lose some of its simplicity when applied to truss panels which differ from the typical one shown in the author's Fig. 1. For example, if the web system has no vertical members at the panel points it is necessary to insert phantom vertical bars at such points unless entirely different deflection equations are to be developed. For a truss with a K-web system, application of the method would seem to be even more difficult.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

LATERAL PILE-LOADING TESTS

Discussion

BY MESSRS. J. C. MEEM, AND T. KENNARD THOMSON

J. C. MEEM,[10] M. AM. SOC. C. E. (by letter).[10a]—Information of undoubted interest and value is presented in this paper. The author has presented the results in such a way as to make them practically applicable. As far as it goes, the paper covers the subject so thoroughly that there is little left to be added in the nature of actual discussion.

The writer feels, however, that in this general field there is a wide opportunity for further research, as applying more particularly to tests made under otherwise similar conditions on piles or foundations that bear on widely varying types of soils, such as clay, loam, sand, gravel, etc. For instance, in his preliminary paragraph of conclusions, the author states: "* * *. The following conclusions, therefore, are confined to the soil conditions and piling arrangements under which the tests were made."

Wherever it is possible to do so, measurements should be made and tabulated of lateral thrust against structures and of resistance to such thrust under conditions similar to those noted. After all, the resistance of the soil to lateral thrust is only secondary in a minor degree, if at all, to that of the pile or structure itself, and conditions exist, or may readily be conceived, in which the instability of the soil is such that tie-backs or lateral bracing must be introduced.

A case in point, which is of paramount interest, is that in which a monolithic concrete pier of a bridge at Grand Coulee Dam, in Washington, was tilted by movement of the glacial drift surrounding it. This movement[11] was such that it reached a maximum of 9 in. at the top, approximately 150 ft above the base.

Such glacial movement could only occur in soils having a predominating clay or equivalent content. It illustrates, however, the necessity for a full understanding of soil conditions where the structure is subject to lateral stresses and, more particularly, where there is a possibility that future operations may tend to change the status of the existing conditions.

NOTE.—The paper by Lawrence B. Feagin, Assoc. M. Am. Soc. C. E., was published in November, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1935, by A. E. Cummings, Assoc. M. Am. Soc. C. E.

[10] Civ. Engr., Brooklyn, N. Y.

[10a] Received by the Secretary December 7, 1935.

[11] *Engineering News-Record*, November 7, 1935.

T. KENNARD THOMSON,[12] M. AM. SOC. C. E. (by letter).[12a]—Experiments are always interesting, but care must be exercised against drawing unwarranted conclusions, especially in subsoils where borings may be inadequate. The author is very wise, therefore, in stating that: "In view of the fact that these tests were conducted in only one type of soil, and in view of the many variables involved, no mathematical analysis is included."

The writer has read many papers supported by beautifully drawn diagrams showing uniform curves of underground pressure, in which regularity might be imagined, but probably never exists. In the sense that no two blades of grass are identical, probably no two cubic feet of subsoil are identical either; conditions created by the material, the pressure exerted by Nature in compacting the mass, variations in the ground-water level, the roots of trees, boulders, etc., are probably different in each case.

Such uniform or symmetrical lines of pressure might possibly occur in man-made piles of sand—but never in foundations on original subsoil.

Many years ago, the writer received a letter from an architect about 300 miles away, containing a long and apparently full description of a proposed foundation, and was asked to telegraph his fee for giving his opinion as to the safe load per square foot permissible to be placed on the foundations. The proposition seemed so plain and simple that the writer was almost misled by it, but suddenly changed his mind and telegraphed that he would not give such an opinion without first examining the site. The architect made no reply, but some time afterward the writer, by mere chance, heard that the "proposed" building had been constructed in another city and had collapsed before the architect had tried to secure the writer's endorsement.

The only time that one can feel really sure about stresses underground is when they are purely hydrostatic—and even then the surrounding conditions must be understood thoroughly. In pneumatic caisson work for buildings in Manhattan, the writer has seldom noticed the air pressure to vary more than a few pounds from the theoretical head, with no allowance for weight of soil, adjoining buildings, etc.; but the horizontal Hudson River Tunnels are different, probably due to the great speed of construction as well as to the closed shields, where sometimes only one-half the anticipated air pressure is required. Slow progress would probably greatly increase the amount of air pressure involved in any case by giving the water above a chance to circulate.

The proportions of sand, clay, boulders, etc., vary in every location even a few feet apart. Sometimes, the engineer finds more than a dozen distinct layers, with different colors of sand and clay in a single vertical foot.

Clay, dry or mixed with sand, boulders, etc., often makes a natural concrete so compact that deep excavations may be made without any side pressures, and, consequently, without sheathing or shoring, but a small quantity of clay in sand under water, often makes the entire mass slide with disastrous results. The author's experiments apply the tests at or near the tops of the piles; but, in actual work, the stresses might occur near the bottom of the piles, or anywhere between the bottom and top, due to quicksand or wet clay in the subsoil stratification.

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[12] Cons. Engr., New York, N. Y.

[12a] Received by the Secretary December 13, 1935.

Some of the reasons for observing the locations (not always made clear in the reports of borings) are to determine the location and character of any adjoining buildings, rivers, lakes, etc., and the heights and character of any adjacent hills. The borings themselves should give the ground-water level and whether or not it varies. Of course, "core" borings are more definite than auger or "wash borings"—but in some cases where the engineer knows, the subsoil conditions at the particular site—"core" borings would simply be a waste of money.

Some years ago, about 12 acres of land dropped 20 ft, sliding down hill and carrying buildings, a 200-ft chimney, etc., with it in a few seconds. A geologist from the South and another from the North had each made twenty-five auger borings, and then the writer was consulted. As he could not find the cause from the borings, he decided to sink a 4-ft shaft, which penetrated good compact material for a depth of about 30 ft. At that depth, a stratum of very sloppy clay, almost soup, was encountered, which explained everything. The site of the factory was surrounded on two or three sides by high hills, and above the factory was a fair-sized lake.

There had recently been a very rainy season and some of the rain had found its way to the clay stratum, which was at the level of the creek at the foot of the slope. The slide resulted, moving the creek itself about 20 or 30 ft. The first day the writer saw the site he advised draining the lake (which was above the most important building left standing) and this was done at once. It had been claimed that a pile of broken stone, covering a comparatively small area was responsible for the damage; but the broken stone did not weigh 1 ton per sq ft on the small area it covered. Furthermore, a section of virgin forest near-by had also sunk.

That stresses may occur at any point along the entire length of a pile is demonstrated by still another case. A railroad company had constructed a high earth-fill leaving an opening for an existing railroad at right angles to its line. This opening was provided for by driving 1 600 wood piles, on top of which were built two side walls, $8\frac{1}{2}$ ft thick, of concrete. These side walls were 33 ft apart and connected at the top by a concrete arch, supporting the new fill, but had no connection at the bottom above the wooden platform on the piles.

When the writer examined the site he found the concrete abutments badly shattered, having moved horizontally 10 ft at one end and 3 ft at the other. The elevation of the original ground was about 1 300 ft above sea level, and the hills near-by were about 700 ft higher.

It was obvious that the water must have found its way from these hills to a clay or quicksand stratum at a level somewhere above the bottom of the foundation piles. In this case the piles may have weakened instead of strengthened the structure. If a circular concrete tunnel had been constructed in the first place, it would not have required as much material as was actually used in the abutments or side walls, and could not have collapsed. It might even have stood without any pile foundation.

As each site requires a special study, an unlimited number of examples might be quoted.

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DISCUSSIONS

DISTRIBUTION OF STRESSES UNDER A FOUNDATION

Discussion

BY MESSRS. W. S. HOUSEL, AND N. M. NEWMARK

W. S. HOUSEL,[40] ASSOC. M. AM. SOC. C. E. (by letter).[40a]—A contribution to the subject of foundation pressures is embodied in this paper, which has immediate value to the practicing engineer and student of soil mechanics. The concise presentation, with complete references, should make it easy for any one interested in the subject to be well informed on the mathematical treatment of a problem which heretofore has been exceedingly difficult to review completely. The solutions of several cases of pressure distribution, although still dealing with conditions more ideal than real, are a closer approximation to actual conditions encountered in practice and should form the basis of enlightened discussion and further development.

The author's analysis of various experimental results is most valuable in calling attention to the fact that practically all previous investigations introduce a variable pressure distribution on the bearing area which controls the maximum pressure ordinate at depths beneath that area. Scarcely less important is the emphasis on the concentration factor which is discussed in connection with solutions. The writer is inclined to feel that the equations presented[40b] are not as convenient as those which have been developed in terms of trigonometric functions, but that does not detract from the value of the analysis and discussion. There is little to be added to the author's use of the tools supplied by the mathematical theory of elasticity and his conclusions mark the point from which discussion may begin. These conclusions are: "(1) The manner in which pressure is distributed over the contact surface must be considered; and (2) the equations of the theory of elasticity must be modified before they can be applied to soils."

In the writer's opinion the only fault to be found with the first conclusion is that it might well be made more emphatic. So much confusion and false

NOTE.—The paper by A. E. Cummings, Assoc. M. Am. Soc. C. E., was published in August, 1935 *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: October, 1935, by Messrs. Clement C. Williams, D. P. Krynine, and L. C. Wilcoxen; November, 1935, by Messrs. Marshall G. Findley, and M. A. Biot; and December, 1935, by Messrs. Jacob Feld, George Paaswell, and G. S. Salter.

[40] Asst. Prof., Civ. Eng., Univ. of Michigan; Research Consultant, Michigan State Highway Dept, Ann Arbor, Mich.

[40a] Received by the Secretary December 6, 1935.

[40b] In Equation (7) of the paper the ratio, $\frac{r_o}{d}$, should be squared.

deduction has arisen from a failure to appreciate the significance of pressure distribution on the bearing area itself that it certainly is gratifying to find this fact recognized as being one of two major conclusions supported by the author's study.

The second conclusion also impresses the writer and, when generally accepted, opens the door to definite progress toward the real problem of how the elastic equations may be altered so that they may be applied to soils. The author suggests the concentration factor, n, as the means of altering the equations and indicates that n must be varied in some manner, depending on the type of material, the load applied whether high or low, and the size of the bearing area whether large or small. With all this there seems to be no room for anything but enthusiastic and unqualified agreement. Thus, the paper establishes the fact that research in foundation pressures in its broadest sense implies much more than a mathematical study of the distribution of load concentrations through bodies of material indefinite in extent. It is also recognized that such mathematical studies have not yet provided foundation engineering with a practical working theory which completely describes the effect of foundation pressures on the soil masses upon which structures must be built.

The writer proposes to discuss the correlation of pressure distribution studies with a practical working theory of soil resistance which may describe soil behavior more completely and which may provide definite methods of measurement by which variable factors not included in the elastic theory may be evaluated. In its fundamental aspects the analysis of soil behavior does not differ from that of any other problem in mechanics of materials. There are two major steps involved in such a problem: First, it is necessary to determine the stresses imposed on the material by application of load; and, second, it becomes necessary to evaluate the resistance of that material to those stresses in terms of deformation and ultimate capacity to carry load. It makes little difference whether the material is in an indefinite mass or an articulated structure, a complete solution of the problems presented cannot be found until both aspects are subjected to analysis.

Pressure distribution based on elastic theory bears only on the first aspect of the major problem and, being limited to materials of assumed ideal properties, has not entirely fulfilled its purposes even there. In the writer's opinion elastic theory has almost, if not entirely, been exhausted in producing results of practical value, and valuable as these results have been they leave a considerable gap to be filled in before foundation problems can be treated on a rational basis. They have given foundation engineers, however, a number of sound conceptions as to how load may be transmitted through soil masses and the type of stresses that would result. The task of securing recognition of the limitations of the elastic theory and the building of new conceptions which may be free from these limitations are fields for further progress. Before attempting further development along this line it seems pertinent to define the attitude with which the writer proposes to approach these problems and to set up certain criteria which may have value in testing the validity of theories which may be proposed. These criteria which represent the atti-

tude of modern physics toward natural phenomena are nowhere more aptly presented than in the definition of a theory given by Dushman[41]. According to this eminent physicist, a theory is nothing more than a system of conceptions which will enable one to describe the results of experience and observation, a system which shall be as complete as possible but, at the same time, simple and comprehensive. Born[42] also serves warning that "the true laws of Nature are relations between magnitudes which must be fundamentally observable. If magnitudes lacking this property occur in our theories it is the symptom of something defective."

Accepting this creed, the development of theories concerned with foundation pressures consists of an orderly process of collecting data by careful observation; the analysis and correlation of these data; and, finally, the formulation of conceptions describing the results of observations only in terms of magnitudes which may be subject to direct observation.

It is not necessary to attempt complete citation of the various experimental and mathematical investigations in the light of the author's excellent review of both. The writer will attempt only to discuss conclusions which may be drawn from these investigations and limitations which must be recognized in applying the conclusions to practical problems.

Distribution of Pressure Beneath the Surface.—From a survey of the results of the various experimental investigations it appears that there are two types of pressure distribution curves which show a significant difference. This difference is illustrated in Fig. 12, in which Curve 1 denotes the kind

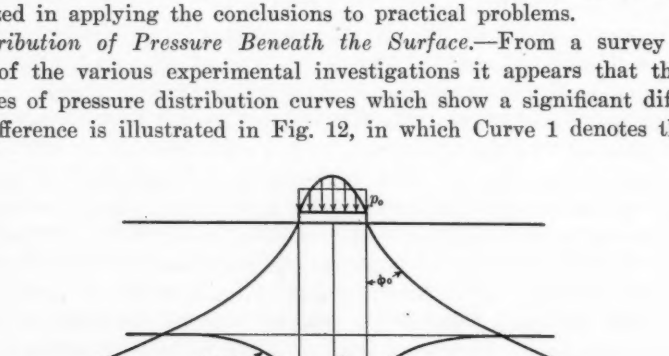

FIG. 12.

of distribution found on horizontal planes relatively close to the bearing area. It yields definite evidence of a concentration of pressure in the region beneath the bearing area, which may be referred to as the central column of the compression cone. This concentration of pressure is a phenomenon that has been noted in practically all experimental studies. It has generally been described in two ways: First, by increased values of the vertical pressure, p_z, in the central column; and, second, by decreased values of the angle of lateral distribution, ϕ_o, for planes close to the bearing area. Curve 2, which is typical of pressure distribution at greater depths where intensities

[41] "Modern Physics—A Survey", by Dr. Saul Dushman, *General Electric Review*. June and July, 1930.

[42] *General Electric Review*, June 30, 1930, p. 332.

of pressure are not excessive, shows no evidence of the foregoing concentration of pressure. Intensities beneath the bearing area appear to be more consistent with those in the compression cone outside the central column and the angle of lateral distribution appears to approach a constant maximum value which, by experiment, has been found to be approximately 60 degrees.

The concentration of pressure in the central column is accompanied by a non-uniform distribution of pressure on the bearing area which the author has demonstrated in his analysis of experimental results. Rather than to state that the non-uniform pressure distribution on the contact surface is the cause of the concentration of pressure in the central column, it seems more logical to conclude that both result from some other phenomenon of primary importance. It is at this point that elastic theory becomes quite inadequate and must be displaced by more basic conceptions not limited by the assumptions of elasticity.

In the first place soils do not possess the properties of the perfectly elastic material necessary to satisfy the very important condition of continuity of stress-strain relationship. The fact that the equations of elasticity yield pressure distribution curves which are at least qualitatively adequate does not alter the fact that the quantitative results are of limited value. This situation arises from the fact that distribution of pressure through soil masses follows laws depending upon statistical regularity which may obtain even in an essentially discontinuous medium.

Failure of the elastic theory appears to originate from two distinctly different sources. First, there are discontinuities arising from boundary conditions which are introduced by the problem of treating load applications over finite areas resting on an indefinite body of material. These considerations are entirely a function of dimensions not peculiar to any particular material and, to that extent, are independent of the physical properties of the material. Second, there are discontinuities arising from the manner in which the material develops resistance to stress or rather fails to develop resistance to stress. The latter phenomenon is affected by dimensions of the loaded area, but is primarily related to properties of the material.

Strohschneider[43] was the first investigator to recognize the significance of this particular characteristic of pressure distribution in soils, and he described pressure distribution as being confined to a cone-shaped region of stress, outside of which there was no apparent effect due to the applied load. He also attempted to alter the elastic equations to produce the concentration of pressure in the central part of the compression cone. His attempt to extend these equations was satisfactory only in special cases, as no single continuous function appeared to be flexible enough to satisfy the wide variation in soil behavior.

In the writer's opinion the concentration factor introduced by later investigators and discussed by the author will not prove to be any more satisfactory in the final analysis. The imperfections in the elastic theory as applied to soils are more deeply seated than that. The discrepancies arise from actual

[43] See "Pressure Distribution in Building Soil", by Dr. F. Kögler and Dr. A. Scheidig, *Die Bautechnik*, No. 29, 1927.

discontinuities in behavior, and it appears fairly obvious that no amount of adjustment will fit these phenomena into a framework based fundamentally on continuity.

The lateral distribution of vertical pressure from a finite area is primarily a boundary phenomenon. This may be illustrated by some results obtained experimentally by Kögler and Scheidig[a]. These results, shown in Fig. 13,

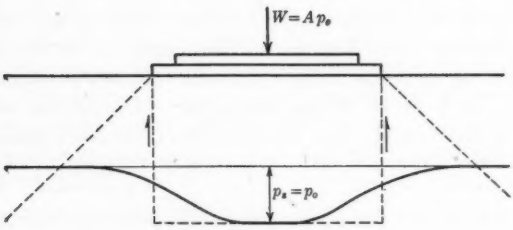

FIG. 13.

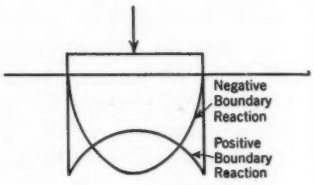

FIG. 14.

represent the pressure distribution obtained at some depth beneath a large rigid plate resting on the surface of a sand fill. The dimensions and rigidity of the plate are such that the load distribution at the contact surface is essentially uniform except for some secondary boundary effects which may be neglected in the present illustration. The important point is that at a depth slightly less than one-fourth the diameter of the loaded area the initial pressure, p_o, was transmitted downward without change in the central region beneath the bearing area. The original pressure distribution is shown assuming a uniform pressure, which is approximately true in this case. The manner in which lateral distribution occurs may be described in simple terms and demonstrates that it is essentially a boundary effect.

If the central column is to be relieved of any of the applied pressure as the load is carried downward this must be accomplished by means of shearing resistance developed on the vertical planes through the edge of the bearing area. The increase in pressure ordinates outside the central column and the decrease in ordinates inside the central column both originate at the boundary and this combined effect extends horizontally in both directions from that plane as the depth increases. As long as the horizontal plane considered is sufficiently close to the bearing surface, the effect from opposite edges does not overlap and the maximum pressure, p_o, may be transmitted downward without change over some width dependent on the limiting angles of influence.

It is also a necessary condition that the areas of the pressure diagram at each depth be equal. The areas taken from the original pressure diagram should be equal to the areas transferred to the region outside the central column. This condition is not entirely fulfilled in the present illustration due to inaccuracy in assuming a uniform distribution of pressure in the absence of more complete information regarding distribution on the contact surface.

[a] "Pressure Distribution in the Soil", by Dr. F. Kögler and Dr. A. Scheidig, *die Bautechnik*, 1927, No. 31, pp. 1–3.

One further point to be brought out in connection with Fig. 13 introduces the resistance of the soil itself as influencing the transmission of pressure. If the loading is continued beyond the point that produces shearing stress on the perimeter planes in excess of the shearing resistance of the soil no more pressure can be transmitted to the region outside the central column. It is obvious then that such additional increments of load must be carried by the central column as a concentration of pressure, a condition which may obtain until the central column is loaded beyond its capacity. Thus several distinct stages of behavior are introduced between which the relations involved are quite dissimilar and, therefore, may not be expressed by a continuous mathematical function.

Distribution of Pressure on the Contact Surface.—The importance of the pressure distribution at the surface of contact has been emphasized by the author and in this discussion. The type of pressure distribution depends primarily on boundary conditions and resistance developed by the soil mass; and it varies depending on the relative rigidity of the bearing plate, the dimensions of the loaded area, and the relative intensity of the load. With all these factors involved, the most fruitful line of attack appears to be to consider types of pressure distribution and their relation to the manner in which resistance to load is developed. It may then be possible to formulate methods of evaluating resistance to load which are dependent on direct measurements and independent of quantitative determination of the actual pressure distribution. This seems to be the only feasible method of approach as it is evidently impracticable in general practice, if not impossible, to measure actual pressure distribution on all foundation substructures.

There are two major types of pressure distribution curves which are illustrated in Fig. 14, and which depend upon the type of boundary reaction developed. In general, materials that are cohesive develop stress concentration at the edge of the bearing area which may be described as a positive boundary reaction. As a result the pressure on the bearing area is maximum at the edge and minimum at the center. In this case the boundary is a source of strength and the reaction developed at that point contributes to the bearing capacity in proportion to the perimeter-area ratio. There are also cases in which the boundary is a source of weakness due to inability to develop resistance to vertical pressure which may be expressed as a negative boundary reaction. The resulting pressure distribution gives maximum ordinates at the center of the bearing area and minimum ordinates at the edge. Such a condition obtains at the surface of bodies of granular material and is typical of most of the experimental investigations of pressure distribution that have been made.

The loss of bearing capacity at the boundary of the loaded area is not difficult to explain by the manner in which load is transmitted through a granular mass. Particles in contact with the bearing plate transmit vertical load to other particles below the surface through points of contact. There being no cohesion the only means of transmission is by arching action, with the reaction to horizontal thrust in the elementary arches being furnished by adjacent particles. At the edge of the area and at the free surface the resist-

ance to horizontal thrust is zero, and the mass is incapable of furnishing resistance to vertical load. In the center of the bearing area horizontal thrusts are partly balanced and, consequently, the elementary soil arches have some stability and are capable of carrying higher intensities of vertical load.

The situation described is not generally encountered in practice although it is representative of the conditions under which most experimental investigations have been performed. Substructures and footings are seldom located on an exposed surface, but are located at some depth with sufficient static head, due to overburden, available to furnish a considerable resistance to lateral displacement. Bearing-capacity tests on granular material conducted under the writer's supervision have shown a positive boundary reaction for the range during which there is sufficient resistance to prevent lateral displacement of the granular material at the edge of the bearing plate[45]. The behavior of foundation mats in general indicates that boundary reactions are more often positive than negative, inasmuch as such mats almost invariably become dish-shaped with the concave surface upward. The smaller settlements at the edge of such mats, over which the loads are fairly well distributed, can only be accounted for when the reaction of the soil mass is greater at that point. The writer has encountered very few cases in which foundation mats settled more at the edge than at the center, and these settlements could be traced definitely to heavy load concentration at the edge of the mat.

Somewhat apart from dimensional effects and soil reactions which affect pressure distribution on the contact surface, consideration must be given to rigidity of the bearing plate. All other things being equal, the pressure distribution may vary from one extreme type to the other, depending upon this rigidity. Flexible bearing plates carrying a distributed load will conform in general to the deformation of the soil mass sufficiently close to develop a uniform pressure on the plane of contact. Concentrated loads on flexible plates are generally transmitted through the plates without fully developing the contact area, resulting in high concentrations of pressures directly under the load.

Pressure distributions on rigid bearing areas are largely controlled by reactions in the soil mass regardless of whether the load applied to the plate is concentrated or distributed. Some of the most effective studies of this type of problem have been made by photo-elastic methods[46], an example of which is shown in Fig. 15. A bakelite sheet, pressed on by a rigid steel die, yields a stress pattern with high concentrations at the edges of the die and a point of minimum stress in the center. Similar experiments with flexible plates show that the concentrations of pressure at the edge disappear, and that the distribution may become practically uniform.

In the preceding discussion the writer has attempted to call attention with some degree of completeness to those factors which enter into a determi-

[45] "Subgrade Soil Testing Methods", *Proceedings*, Am. Soc. for Testing Materials, Vol. 34, Pt. II, pp. 730–737.

[46] "Kinematography in Photoelasticity", by M. M. Frocht, *Transactions*, Am. Soc. M. E., Vol. 54, No. 11, p. 83.

nation of stresses in a foundation. Most, if not all, of these factors are fairly well recognized although they may be described in different terms. The attempt has been made in this discussion to place emphasis on certain

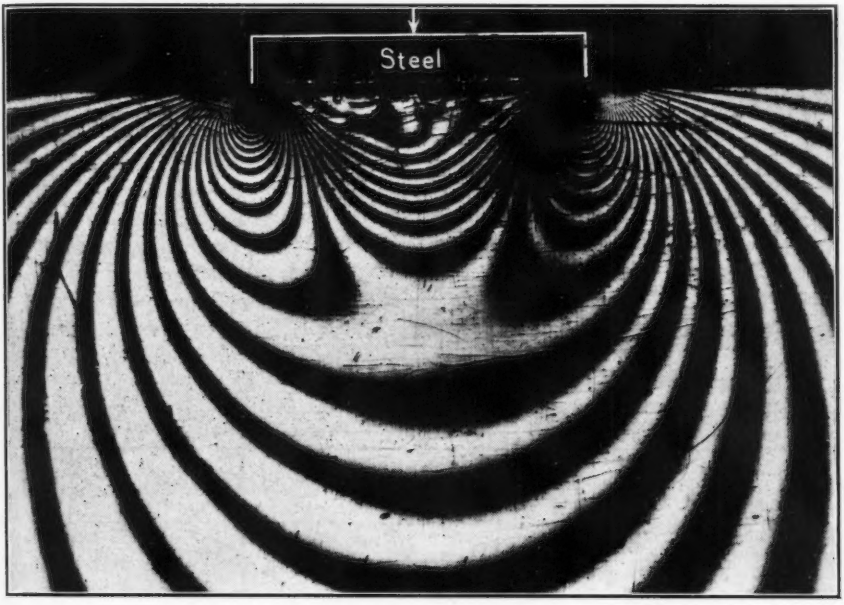

FIG 15.—STRESS DISTRIBUTION BY PHOTO-ELASTICITY.

aspects of pressure distribution which the writer believes may be correlated and built into a system of conceptions that will serve as a practical theory of soil resistance.

Because of the fact that some of the assumptions of elastic theory cannot be applied to soils, the distribution of stress predicated on elasticity cannot be considered satisfactory except in a qualitative sense. These limitations of elastic theory lead one to the conclusion that it is essential to simplify the methods of dealing with the problem, thus making it possible to include some important factors that otherwise may be ignored. It seems entirely logical in a qualitative treatment of pressure distribution to accept such simplification as long as the results are only used to establish other relations which may be determined on a proper quantitative basis. This procedure is particularly necessary in order to describe pressure distribution in regions close to the bearing area where dimensional effects and discontinuities due to several different stages of soil resistance are most critical.

A Simplified Method of Pressure Distribution.—Such a simplified method of pressure distribution is shown in Fig. 16, in which it is proposed to deal with pressure distribution by linear relations combining triangular and rectangular pressure distribution. All the examples in the succeeding discussion will be treated in terms of the two-dimensional problem for a bearing

area of finite width, b, and indefinite length. The solution of this problem based on the elastic theory is credited to Michell and has been reviewed[47] by George Paaswell, M. Am. Soc. C. E. The problem will be treated in terms of a uniform pressure, p_o, at the contact surface and the effect of a non-uniform

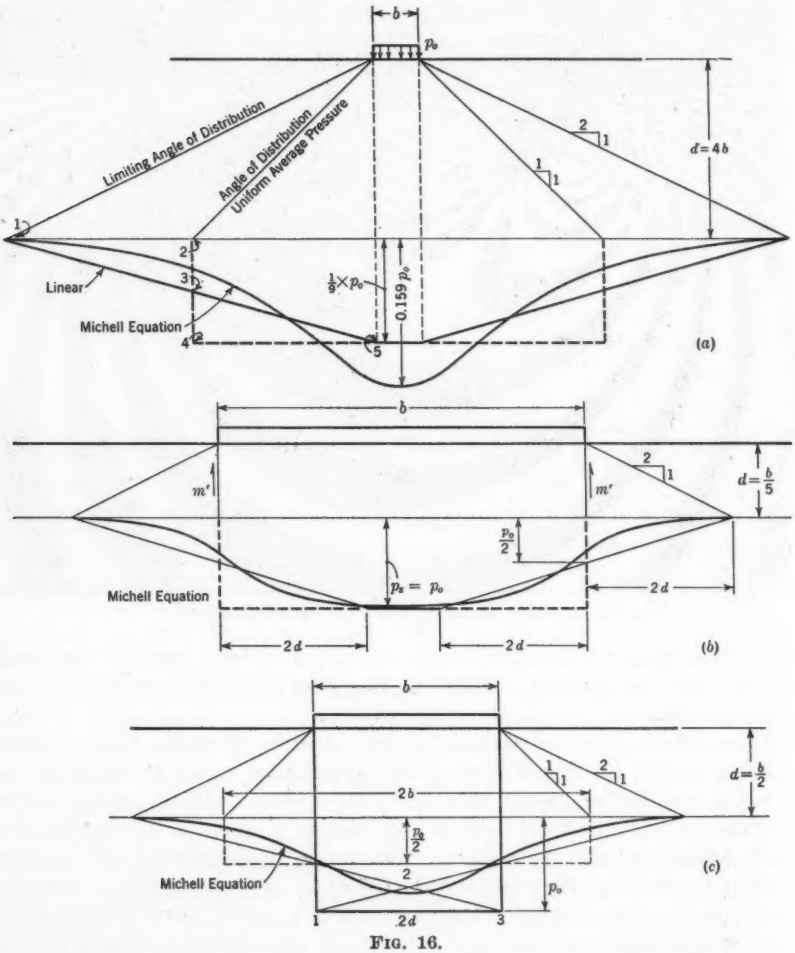

FIG. 16.

distribution discussed as a separate consideration. The transition from the pressure diagram for the bearing area to the pressure diagram on any horizontal plane is accompanied by satisfying the condition that the area of the respective diagrams is equal, as shown in Fig. 16(b).

In Fig. 16(a) the distribution curve as given by the Michell equation is replaced by a straight line which varies from zero at the limiting angle of

[47] "Transmission of Pressure Through Solids and Soils and the Related Engineering Phenomena", by George Paaswell, *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 1563.

distribution with a slope of 1 on 2 to a maximum ordinate at the vertical plane through the edge of the bearing area. A uniform pressure or rectangular distribution is used under the central column of the compression cone. An average or uniform pressure commonly assumed in practical problems is obtained by a redistribution or juxtaposition of equal triangles, 1-2-3 and 3-4-5. In order that the areas of the pressure diagram shall be the same, the angle of average pressure distribution must be 45° or a 1 on 1 slope which is a value commonly assumed. The proper value of the limiting angle of distribution is an important factor and is taken as a slope of 1 on 2 or 63° 26′ for several reasons. In the first place the experimental investigations have indicated an angle of about 60 degrees. It is improbable that the very small pressures near the limiting zone of influence would be recorded by the measuring devices thus far used; therefore, the difference of 3.5° is not significant. As will be shown subsequently the relation between the angle of average pressure distribution and the limiting angle of influence verifies the slope of 1 on 2 since the equilibrium of elements within the compression cone requires that the average angle of distribution be 45 degrees.

For the purpose of discussion accepting the assumption that the distribution angles are correct, three cases of pressure distribution arise, defined by the ratio existing between the width of the bearing area and the depth to the horizontal plane considered. First, if the depth is less than $\frac{b}{4}$ as in Fig. 16(*b*) the full intensity of the applied pressure reaches the horizontal plane over the middle of the central column. Such a distribution takes into account the fact that the lateral distribution is a boundary effect which depends upon the shearing resistance, m', acting on the vertical planes through the edge of the bearing area. The result is also in conformity with results of Kögler and Scheidig shown in Fig. 13. Second, a special case exists in which the depth, d, is equal to $\frac{b}{2}$. As illustrated in Fig. 16(*c*) the same principles are applied in obtaining the pressure distribution as were used in the previous example. Pressure triangles are taken out of the original pressure diagram equal to the pressure distributed on either side of the central column. The triangle, 1-2-3, is subtracted twice, being included in the distribution to both sides and reducing the ordinate of the rectangular part of the diagram to $\frac{P_0}{2}$. The average pressure distribution may be used to obtain the same result, considering the average pressure within a 45° angle of distribution, and changing from the rectangular to the triangular distribution outside the central column in the same manner as indicated in Fig. 16(*a*). Third, in all cases when the ratio is greater than $\frac{b}{2}$ it is necessary first to obtain the average pressure to determine the center ordinate as shown in the diagram, as it is impossible to work directly from the original pressure diagram as illustrated in Fig. 16(*b*).

The comparison between the simplified pressure distribution and the elastic solution by Michell has been made for the purpose of correlating the proposed method with previous investigations. In the light of the limitations of the elastic theory, the writer does not feel, however, that the mathematical solution is entitled to any pre-eminence as a criterion of accuracy as far as quantitative results are concerned. Nevertheless, it is interesting to note wherein the two methods are at variance and, in the light of all factors that should be considered, to discuss the probable accuracy of the results obtained.

For the case in which the depth considered is less than $\frac{b}{4}$, the agreement is quite close, but the Michell solution fails to give a center ordinate equal to p_o which should be the case if the boundary effects are properly treated. As soon as the boundary effect from either edge overlaps, as for depths greater than $\frac{b}{4}$, the elastic solution gives a center ordinate greater than that obtained from the simplified method. In view of the factors involved, the writer can see little choice in either method in judging the accuracy of the result. The simplified method appears to include the more rational treatment of boundary effects; but at best it is only an approximation. In all cases the elastic solution gives smaller ordinates in the regions more remote from the central column.

When non-uniform distribution on the contact surface is considered in relation to the phenomenon of lateral distribution of pressure the results of the simplified method appear more favorably. In that range of load where shearing resistance on the boundary planes has not been exceeded, distribution on the contact surface would tend toward maximum pressures at the edge and minimum pressures at the center. Translating these conditions into pressure distribution on planes below the bearing area, it is apparent that center ordinates would be decreased and ordinates outside the central column increased in closer conformity with the simplified pressure diagrams. It is felt that this is the case more commonly encountered in practice.

If the pressure distribution on the contact surface is maximum at the center and minimum at the edge as in most experimental results available, this behavior may be traced to a failure to affect lateral distribution at the boundary, and there is a consequent concentration of pressure in the central column. In this case the simplified method does have a real advantage because it affords a simple and logical method of treating concentration of pressure in the central column. The importance of this advantage lies not so much in the determination of pressure ordinates in the central column as it does in the correlation of stress distribution beneath a loaded area with ability to carry load. After all, the determination of stresses in the soil mass is merely the first step in developing methods of evaluating bearing capacity. Thus, from a study of pressure distribution, it has been found that one of the two major types of stress is shown to be concentration of pressure in the central column, and the other, to be resistance developed on the boundary planes of the central column which affects a lateral distribution of vertical pressure.

The problem now becomes one of evaluating the resistance of the soil mass to these two types of stress in terms that may be subject to direct measurement. In Fig. 17 are shown the stress conditions in the compression cone in terms of which it is possible to study the equilibrium of the various elements and establish their ultimate resistance to load.

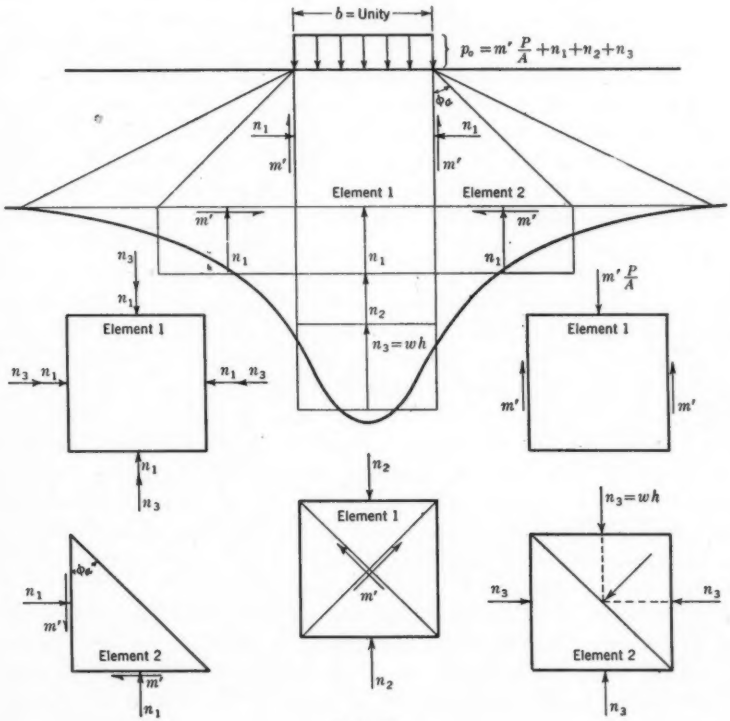

FIG. 17.

Element 1, taken from the central column, offers resistance to deformation by two distinct stress reactions. The first is the summation of three similar pressure components: n_1, the vertical pressure representing cubical compression with lateral components furnished by the confining influence of the surrounding body of material and involving no shearing stresses within the element; n_2, the vertical pressure transmitted downward without increase in lateral components up to the full capacity of the soil to develop shearing resistance on 45° planes of maximum shear; and n_3, a third increment of vertical pressure arising from the static pressure due to surcharge, which in some cases may be available as additional confining influence. All three of these factors are included in a stress reaction which may be defined as developed pressure and represents the capacity of the central column to transmit vertical load directly downward without failure due to lateral displacement.

The second stress reaction must be considered because of the lateral distribution of pressure through the shearing resistance, m', acting on the boundary planes. The vertical load carried in this fashion is a function of the perimeter of the area and may be expressed in terms of the bearing pressure on the area by dividing the total boundary force, $m'P$, by the area, A, as shown in the equation for total bearing capacity, p_0. The equilibrium of Element 2 within the compression cone and outside the central column establishes certain conditions which must be satisfied and leads to some significant relations. As long as the shearing resistance, m', is not exceeded, no concentration of pressure will develop within the central column and the pressure distribution across the compression cone may be represented as uniform pressure, n_1. Element 2 must also furnish the lateral component, n_1, resulting in a shearing stress, m', on the horizontal plane bounding the element. Under the condition that the normal pressure, n_1, acting on both the horizontal and vertical planes of Element 2 are equal, the shearing resistance, m', on both planes must also be equal to n_1. In the first place this condition requires that the angle of average distribution be 45° as no other value of the angle will satisfy all three equations of equilibrium, which require that the summation of horizontal and vertical forces and the summation of moments be zero. The quality of shearing stress on planes at right angles to each other, as stated in Cauchy's theorem[48], also may be used to define the angle, ϕ_a, as having a specific value of 45 degrees. Thus, all the requirements for equilibrium of elements in a body of material are satisfied except the commonly accepted condition of continuity of stress. In the problem under consideration the discontinuity of stress introduced at the boundary of the compression cone is an agreement with experimental observations, particularly the cone-shaped region of stress noted by Strohschneider.[43]

The increments of pressure transmitted as concentrations in the central column do not affect the lateral distribution, as this path of force transmission is cut off when the ultimate shearing resistance of the soil is exceeded on the perimeter planes. The supporting power of the soil is not exhausted, however, until the capacity of the mass to develop pressure is exceeded. Thus, the concentration of pressure in the central column is possible, resulting in a pressure distribution that cannot be represented by an average pressure for the full width of the compression cone. Any attempt to use such an average pressure distribution results in an erroneous value of the angle of distribution. It is important to note that apparent values of this angle of less than 45° are due to a changing proportion of two separate and dissimilar stress reactions rather than to a change in the angle of lateral distribution.

Correlation of Pressure Distribution and Bearing Capacity.—With a complete understanding of the nature of the stresses imposed on a body of soil by foundation loads, it is possible to correlate these stresses with the resistance developed by the soil and to isolate those factors which must be measured in order to evaluate the bearing value of the soil. It is apparent from the review of pressure distribution that two separate stress reactions are involved in the problem and these may be defined as developed pressure and perimeter

[48] "The Mathematical Theory of Elasticity", by A. E. H. Love, pp. 74 and 79.

shear. The bearing capacity, which is the combined effect of these two types of resistance, may be expressed by the following straight-line equation developed in connection with Fig. 17 which is valid for constant value of a third variable, settlement:

$$p = \frac{mP}{A + n} \quad \text{.....(26)}$$

in which p = bearing pressure, in pounds per square foot; m = perimeter shear, in pounds per linear foot; P = perimeter, in feet; A = area, in square feet; and, n = developed pressure, in pounds per square foot.

The developed pressure, n, is the summation of all the increments of pressure developed by the soil mass and designated as n_1, n_2, and n_3 in the previous discussion. For the purpose of interpreting load tests on soil, it is not necessary to measure the separate effects although the subdivision is useful in some other respects. The developed pressure is the source of the major part of the bearing capacity in practical design, the effect of perimeter shear being small for large bearing areas with small perimeter-area ratios.

The perimeter shear is an important factor, however, in the usual sizes of bearing areas for testing purposes and, unless it is properly evaluated, load tests cannot be used to determine the supporting value of large foundations. The perimeter shear, m, is expressed in terms of a concentrated force per unit length of the perimeter because the boundary effects, either positive or negative, involve irregularities in the distribution of shearing stress on the boundary planes which have not been subjected to analysis.

Evidently, there are two types of behavior involved due to boundary stresses. The primary effect of the shearing stress on the perimeter planes is in the lateral distribution of the pressure which controls the depth affected by the surface load. The larger the area the greater the depth required for distribution and, consequently, the greater the settlement for any given intensity of bearing pressure. This leads to the familiar statement that the settlement for a given intensity of pressure is proportional to the diameter of the bearing area. Conversely stated, it may be said that for a constant settlement the bearing pressure is inversely proportional to the diameter, or that it is proportional to the perimeter-area ratio. This statement is correct in so far as one type of reaction is concerned, but it ignores the fact that developed pressure is an important factor in the bearing capacity and it is to some degree independent of the stress reaction at the boundary.

In addition to the primary boundary stresses, there are secondary stress reactions which lead to the non-uniform pressure on the bearing area. These secondary stress concentrations appear to be independent of the width of the area and are rather a function of rigidity of the bearing area, or as in granular materials, are controlled by a lack of resistance to lateral displacement. It appears impossible at present to measure these boundary stresses except in terms of a concentrated force acting at the perimeter and being the integrated result of the complex reactions which have been recognized.

The analysis of a series of bearing-capacity tests has shown that the straight-line equation does reproduce the load-settlement relations for differ-

ent sizes of bearing area to a satisfactory degree of accuracy within a range of size for which the secondary boundary effects are similar. However, due to a change in boundary conditions, load tests of bearing areas of somewhat less than 1 sq ft do not conform to the linear equation for larger plates. Although the reasons for this discontinuity are not completely exposed by present data it appears to be due to the secondary boundary reaction which is independent of the width of the bearing area for the larger test sizes. In the smaller sizes the secondary boundary effects from opposite sides of the area overlap, and failure takes place due to lateral displacement before the full supporting value of the soil can be developed. In both cases, however, the only factor affected is the boundary stress or perimeter shear, and the analysis of tests indicates that the developed pressure remains the same for all sizes of bearing areas. Until further investigation has been made, it appears desirable to avoid the discontinuities referred to and this may be done by the proper selection of test sizes. When this procedure is followed, the straight-line equation is valid and furnishes a reliable determination of soil resistance.

Conclusion.—In this discussion limited space does not permit of a more complete elaboration of the conceptions of soil resistance herein presented, and particularly a general relation which correlates bearing capacity with settlement of foundations. Such a general equation has been developed with examples in which the relations were applied to the actual design of substructures.[49] Examples of a number of series of load tests are available in which the factors of soil resistance have been evaluated successfully[50]. Some progress has been made in developing a method of measuring the resistance of cohesive soils and correlating the cohesion or shearing resistance with the concentration of pressure beneath a loaded area, which has been emphasized in the paper[51].

Progress in foundation engineering or, more exactly, soil mechanics has been rapid in the last few years and important contributions to the science have been made by a number of investigators working in widely separated localities. The most difficult problem to-day, in all this development, is for the investigators themselves, to say nothing of others who are interested, to find an apportunity to engage in a free exchange of ideas which is essential for real progress. It is seldom that a paper as comprehensive and thorough as that presented by the author is available for discussion, and in providing this opportunity Mr. Cummings has made a contribution of double value to the profession.

N. M. NEWMARK,[52] JUN. AM. SOC. C. E. (by letter).[52a]—The close agreement between the wide range of experimental data gathered by Mr. Cummings and the hypothesis of stress distribution in sand attributed to Professors Griffith and Fröhlich is quite remarkable. It seems that the experimental data are

[49] "Bearing Value of Clay is Determinable", *Engineering News-Record*, February 23, 1933, p. 244.

[50] "Bearing Capacity of Clay", *Engineering News-Record*, September 19, 1935, p. 408.

[51] "A Penetration Method of Measuring Soil Resistance", *Proceedings*, Am. Soc. for Testing Materials, Vol. 35, 1935.

[52] Research Asst. in Civ. Eng., Coll. of Eng., Univ. of Illinois, Urbana, Ill.

[52a] Received by the Secretary December 7, 1935.

limited to measurements at depths of less than 5 ft, and in all but a few cases at depths of 3 ft or less. It appears reasonable that, at great depths, sand should act more nearly in the manner of an elastic and homogeneous material than at shallow depths. The experimental data cited, for example, give, for a ratio of depth to radius of plate of approximately 9, and for depths of 60 in. and 36 in., the respective values of pressure as 2.5% and 3.5% of the intensity of load applied. Thus, it seems that n possibly decreases with the depth.

It does not appear to be convenient to consider a varying value of n in the equations given by the author. However, other empirical relations may be found by which it is possible to take into account the change in properties of the sand with depth and also to account for the proper effect near the surface.

A simple method which the writer has devised seems to offer certain advantages. It consists, briefly, in transforming the depth of the given material in an arbitrary manner, the transformation depending only on the depth, and then applying Boussinesq's equations to find the vertical stress on horizontal planes of the transformed material. The vertical stresses thus obtained are taken as acting at the corresponding untransformed depth and will be in statical equilibrium. Statically possible stresses in other directions may be determined as well, but it is not necessary to discuss this matter herein.

Let the depth to a given horizontal plane in the given material be denoted by z, and the depth after transformation by h. Assume that each element of depth, dz, is to be transformed by the relation,

$$dh = \psi\, dz \qquad (27)$$

in which ψ is a function of the depth, z, alone. Then the transformed depth, h, is given by the equation, $h = \int_0^z \psi\, dz$. When ψ is a constant, the transformed depth is, $h = \psi z$.

The value of ψ may be chosen in such a way that the stress at various depths along the line of action of a single concentrated force will be the same as the stress computed for a particular value of n. This value of ψ is found to be,

$$\psi = \sqrt{\frac{3}{n}} \qquad (28)$$

In general, the stress surface for a particular value of n and that for a value of ψ given by Equation (28) will not be the same. However, the difference will be very small, as it apparently must be, since the volumes under each of the stress surfaces are equal, the center heights are equal, and the shapes are somewhat similar. For example, the curves in Fig. 18 show vertical sections through the stress surfaces for $n = 3$, $n = 6$, and for a constant value of ψ equal to $\sqrt{0.5} = 0.707$.

Computations for varying values of ψ are no more difficult than for constant values. Furthermore, all existing tables, coefficients, and formulas for vertical stress due to various given loadings may be used directly merely

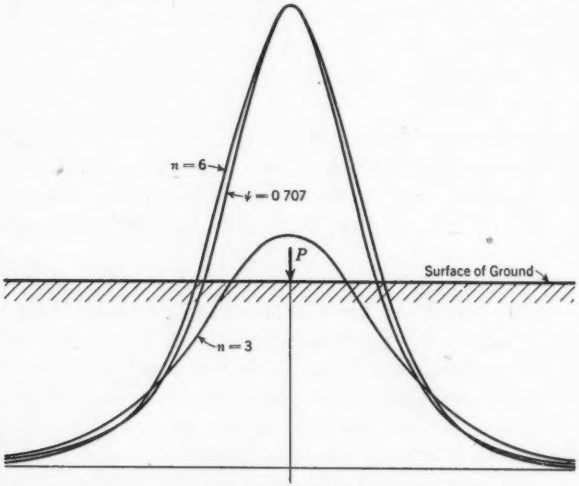

FIG. 18.—RELATIVE FORMS OF STRESS SURFACE.

by taking the value of the transformed depth, h, instead of the actual depth, z. The entire process is essentially the same as a change in the scale of depth, other conditions remaining the same.

Close agreement with the experiments cited by the author will also be obtained by use of a transformed depth. For example, in Fig. 4, curves corresponding to Cases II and IV may be obtained from Cases I and III by transforming the vertical scale of these curves by multiplying by the quantity, $\frac{1}{\psi} = \sqrt{2}$. These curves will lie very close to Curves II and IV, and, consequently, will fit the experimental data as well.

Approximate solutions of problems similar to those discussed by the author may also be obtained by various depth transformations. Pressure distribution through sand to clay, or through alternate layers of sand and clay, might possibly be treated. Of course, the proper method of depth transformation must be chosen, and, apparently, there is very little experimental data upon which to base a choice.